Discrete Mathematics-3140708

Unit-1: Set Theory

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Contents

1 Set theory

1.1 Basic of set theory

Definition: Set

A set is a well defined collection of objects.

Usually denoted by capital letters A, B, C, X, Y, Z, etc.

There are two methods of representing a set :

1. Roster or tabular form

2. Set-builder form.

For example: Write the solution set of the equation $x^2 + x2 = 0$.

Roster form: $\{1, \ 2\}$.

Set builder form: $\{x$: x is roots of $x^2 + x2 = 0\}$

We give below a few more examples of sets used particularly in mathematics, viz.

N : the set of all natural numbers

Z : the set of all integers

Q : the set of all rational numbers

R : the set of real numbers

Z^+ : the set of positive integers

Q^+ : the set of positive rational numbers, and

R^+: the set of positive real numbers.

Definition: Universal set

A set is called a universal set if it includes every set under discussion. Which is denoted by U.

Definition: Empty set or Null set

A set which does not contain any element is called the empty set or the null set or the void set.

For example:

Let A $= \{x : 1 < x < 2, x$ is a natural number$\}$. Then A is the empty set, because there is no natural number between 1 and 2.

Definition: Finite and Infinite Sets

A set which is empty or consists of a definite number of elements is called finite otherwise, the set is called infinite.

For example:

Let W be the set of the days of the week. Then W is finite.

Let G be the set of points on a line. Then G is infinite.

Note: All infinite sets cannot be described in the roster form. For example, the set of real numbers cannot be described in this form, because the elements of this set do not follow any particular pattern.

Two sets A and B are said to be equal if they have exactly the same elements and we write $A = B$. Otherwise, the sets are said to be unequal and we write $A \neq B$.

For example:

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 1, 4, 2\}$. Then $A = B$.

Note:A set does not change if one or more elements of the set are repeated. For example, the sets $A = \{1, 2, 3\}$ and $B = \{2, 2, 1, 3, 3\}$ are equal, since each element of A is in B and vice-versa. That is why we generally do not repeat any element in describing a set.

Definition: Subsets

A set A is said to be a subset of a set B if every element of A is also an element of B.

The symbol $\subset$ stands for "is a subset of" or "is contained in".

$\rightarrow A \subset B$ and $B \subset A \Leftrightarrow A = B$.

For example:

The set Q of rational numbers is a subset of the set R of real numbers, and we write $Q \subset R$.

Note: Let A and B be two sets. If $A \subset B$ and $A \neq B$, then A is called a proper subset of B and B is called superset of A. For example, $A = 1, 2, 3$ is a proper subset of $B = 1, 2, 3, 4$.If a set A has only one element, we call it a singleton set. Thus, $\{a\}$ is a singleton set.

Intervals as subsets of R:

Let a, $b \in R$ and $a < b$. Then the set of real numbers $\{y : a < y < b\}$ is called an open interval and is denoted by (a, b).

The interval which contains the end points also is called closed interval and is denoted by $[a, b]$. Thus $[a, b] = \{x : a \leq x \leq b\}$

We can also have intervals closed at one end and open at the other, i.e.,

$[a, b) = \{x : a \leq x < b\}$ is an open interval from a to b, including a but excluding b.

$(a, b] = x : a < x \leq b$ is an open interval from a to b including b but excluding a.

Definition: Power Set

The collection of all subsets of a set A is called the power set of A. It is denoted by $P(A)$. In $P(A)$, every element is a set.

In general, if A is a set with $n(A) = m$, then it can be shown that $n[P(A)] = 2^m$.

For example: if $A = \{1, 2\}$, then $P(A) = \{f, \{1\}, \{2\}, \{1, 2\}\}$ Also, note that $n[P(A)] = 4 = 2^2$

1.2 Venn Diagrams:

Most of the relationships between sets can be represented by means of diagrams which are known as Venn diagrams.

For example:

In Figure 1, $U = \{1, 2, 3, ..., 10\}$ is the universal set of which $A = \{2, 4, 6, 8, 10\}$ and $B = \{4, 6\}$ are subsets, and also $B \subset A$.

Operations on Sets:

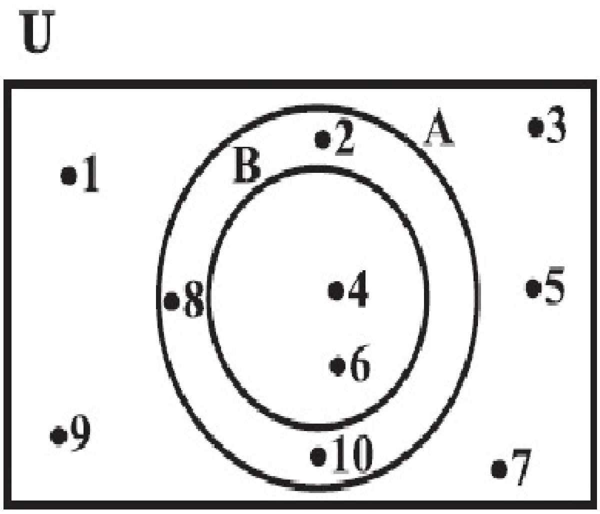

Figure 1: Venn Digram

Definition: Union of sets

The union of two sets A and B is the set $\cup$ which consists of all those elements which are either in A or in B (including those which are in both). In symbols, we write.

$A \cup B = \{x : x \in A\, or\, x \in B\}$.

The union of two sets can be represented by a Venn diagram as shown in Figure 2.

The shaded portion in Figure 2 represents $A \cup B$.

For example:

Let $A = \{2, 4, 6, 8\}$ and $B = \{6, 8, 10, 12\}$ then $A \cup B = \{2, 4, 6, 8, 10, 12\}$

Definition: Intersection of sets

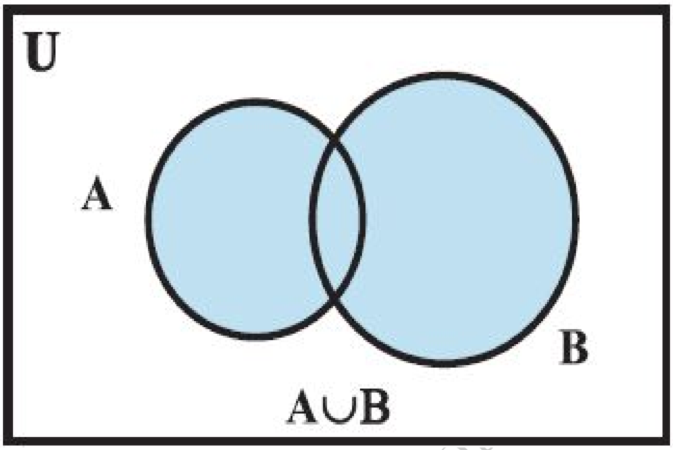

Figure 2: Union

The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write $A \cap B = \{x : x \in A\, and\, x \in B\}$

The shaded portion in Figure 3 indicates the intersection of A and B.

Note: If A and B are two sets such that $A \cap B = \phi$, then A and B are called disjoint

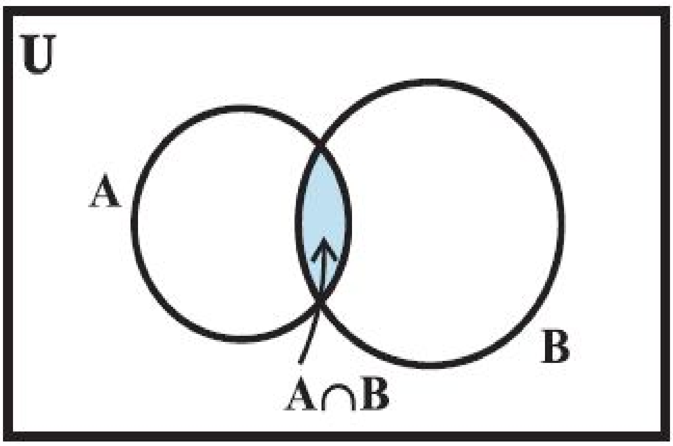

Figure 3: Intersection

sets.

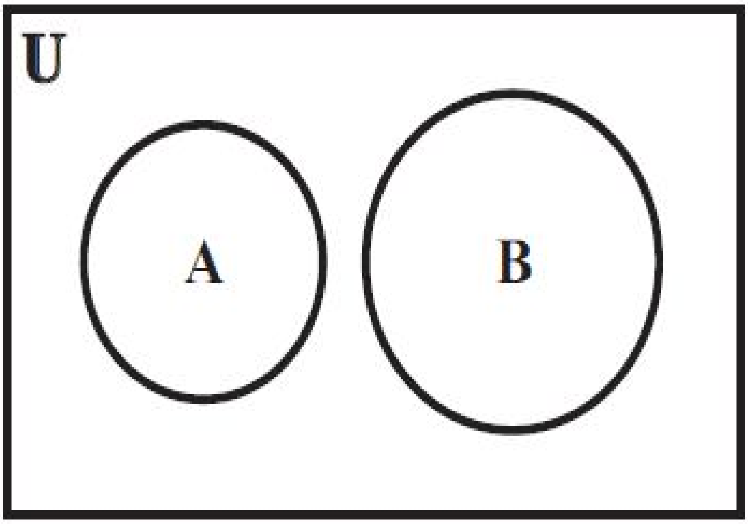

Figure 4: Disjoint sets

Difference of sets

The difference of the sets A and B in this order is the set of elements which belong to A but not to B. Symbolically, we write $A - B$ and read as A minus B.

For example:

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8\}$ then $A - B = \{1, 3, 5\}$ and $B - A = \{8\}$.

Definition: Complement of a Set

Let U be the universal set and A a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. Symbolically, we write A' to denote the complement of A with respect to U. Thus, $A' = \{x : x \in U and x \in A\}$. Obviously $A' = U - A$.

Definition: Cartesian Products of Sets

Given two non-empty sets P and Q. The cartesian product is the set of all ordered pairs of elements from P and Q, i.e., $P \times Q = \{(p, q) : p \in P, q \in Q\}$

If either P or Q is the null set, then $P \times Q$ will also be empty set, i.e., $P \times Q = \phi$.

For example:

A is a set of 2 colours and B is a set of 3 objects, i.e., $A = \{\text{red}, \text{blue}\}$ and $B = \{b, c, s\}$, where b, c and s represent a particular bag, coat and shirt, respectively.

How many pairs of coloured objects can be made from these two sets?

Proceeding in a very orderly manner, we can see that there will be 6 distinct pairs as given below:

(red, b), (red, c), (red, s), (blue, b), (blue, c), (blue, s).

1.3 Some Basic Set Identities

A, B and C are sets, and we consider them to be subsets of a universal set U. Remember that ϕ is the empty set, and that A' means 'the complement' of A.

Commutative Laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative Laws::

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive Laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Identity Laws:

$$A \cup \phi = A$$
$$A \cap U = A$$

Complement Laws:

$$A \cup A' = U$$
$$A \cap A' = \phi$$

Double Complement Laws:

$$(A')' = A$$

Idempotent Laws:

$$A \cup A = A$$
$$A \cap A = A$$

Universal Bound Laws:

$$A \cup U = U$$

$$A \cap \phi = \phi$$

De Morgans Laws:

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Set Difference Law:

$$A - B = A \cap B'$$

Complements of U and ϕ:

$$U' = \phi$$

$$\phi' = U$$

2 Functions

Definition: Functions

A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.

In other words, a function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f. If f (a) = b, we say that b is the **image** of a and a is a **preimage** of b. The **range**, or **image**, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.

The function f from A to B is denoted by $f : A \to B$.

Definition: Real Valued Functions A function which has either R or one of its subsets as its range is called a real valued function. Further, if its domain is also either R or a subset of R, it is called a real function.

For example:

Let N be the set of natural numbers, then a real valued function can be defined as $f : N \to N$ by $f(x) = 2x + 1$.

2.1 Special functions

Definition: Identity function

Let R be the set of real numbers. Define the real valued function $f : R \to R$ by $y = f(x) = x$ for each $x \in R$. Such a function is called the identity function. Here the domain and range of f are R. The graph is a straight line as shown in Figure 6. It passes through the origin.

Definition: Constant function

Constant function Define the function $f : R \to R$ by $y = f(x) = c$, $x \in R$ where c is a constant and each $x \in R$. Here domain of f is R and its range is c (Fig. 7).

Definition: Polynomial function

A function $f : R \to R$ is said to be polynomial function if for each x in R, $y = f(x) =$

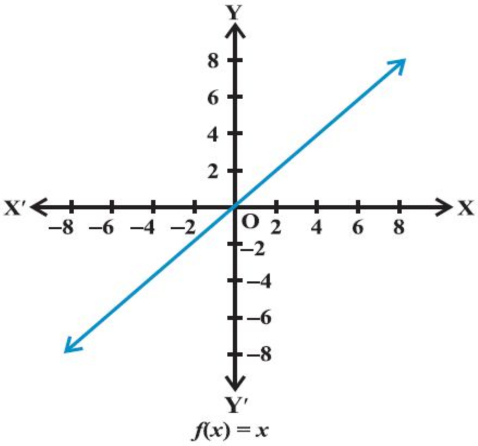

Figure 5: Identity function

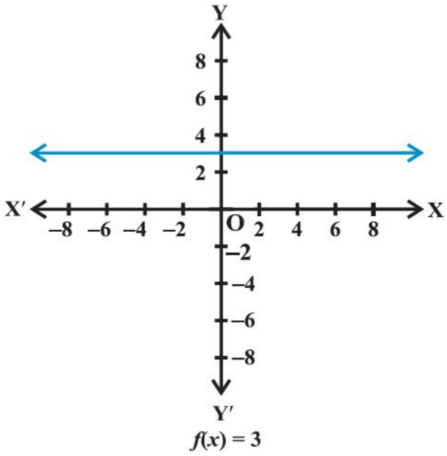

Figure 6: Constant function

$a0 + a1x + a2x2 + ... + anxn$, where n is a non-negative integer and $a0, a1, a2, ..., an \in R$.

The functions defined by $f(x) = x^3x^2 + 2$, and $g(x) = x^4 + \sqrt{2}x$ are some examples.

Definition: Modulus function

The function $f : R \to R$ defined by $f(x) = |x|$ for each $x \in R$ is called modulus function.

For each non-negative value of x, $f(x)$ is equal to x (Fig. 8).

Definition: Signum function

The function $f : R \to R$ defined by

$$f(x) = 1, \ if x > 0,$$
$$= 0, \ if x = 0,$$
$$= -1, \ if x < 0,$$

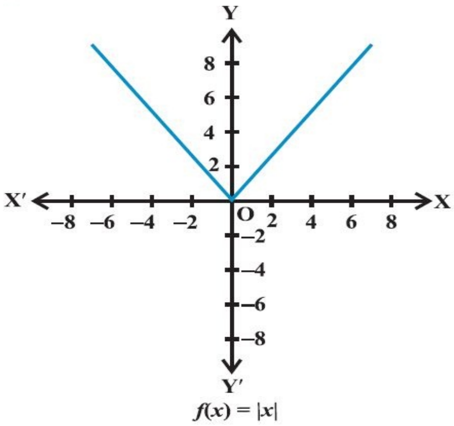

Figure 7: Modulus function

is called the signum function. The domain of the signum function is R and the range is the set $\{1, 0, 1\}$. The graph of the signum function is given by the Fig. 9.

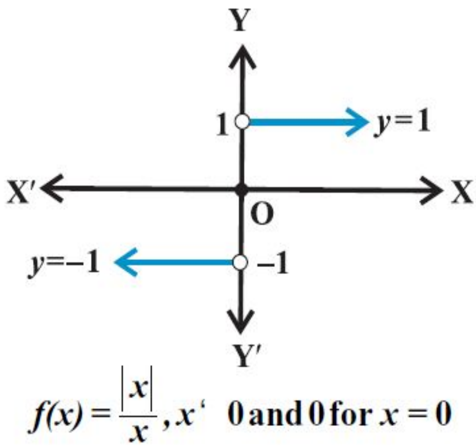

Figure 8: Signum function

Definition: Greatest integer function (floor)

The function $f : R \to R$ defined by $f(x) = \lfloor x \rfloor$, $x \in R$ assumes the value of the greatest integer, less than or equal to x. Such a function is called the greatest integer function. From the definition of $\lfloor x \rfloor$, we can see that

$$\lfloor x \rfloor = -1 \, for \, -1 \leq x < 0, \quad \lfloor x \rfloor = 0 \, for \, 0 \leq x < 1,$$

$$\lfloor x \rfloor = 1 \, for \, 1 \leq x < 2, \quad \lfloor x \rfloor = 2 \, for \, 2 \leq x < 3, \text{ and so on.}$$

The graph of the function is shown in Fig. 10.

Definition: Ceiling function

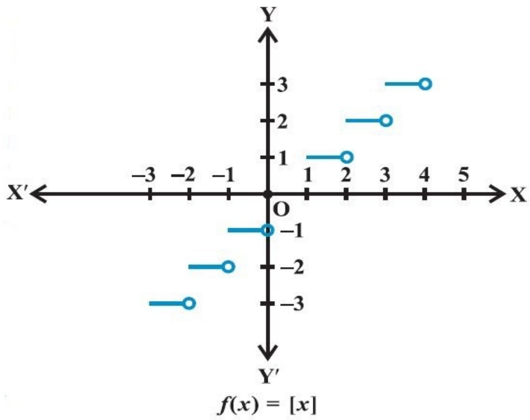

Figure 9: Greatest integer function (floor)

The function $f : R \to R$ defined by $f(x) = \lceil x \rceil$, $x \in R$ assumes the value of the greatest integer, greater than or equal to x. Such a function is called the ceiling function.
From the definition of $\lceil x \rceil$, we can see that

$$\lceil x \rceil = 0 \; for \; -1 < x \leq 0, \quad \lceil x \rceil = 1 \; for \; 0 < x \leq 1,$$
$$\lceil x \rceil = 2 \; for \; 1 < x \leq 2, \quad \lceil x \rceil = 3 \; for \; 2 < x \leq 3, \text{ and so on.}$$

Example

Define the function $f : R \to R$ by $y = f(x) = x^2$, $x \in R$. What is the domain and range of this function? Draw the graph of f.

Solution;

Domain of $f = x : x \in R$.

Range of $f = x^2 : x \in R$.

The graph of f is given by Fig. 13

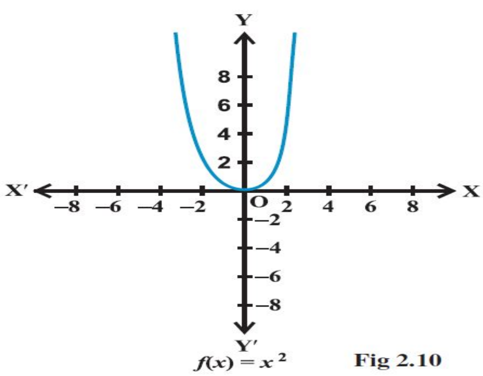

Figure 10: Graph of $f(x) = x^2$

Example

Define the real valued function $f : R - \{0\} \rightarrow R$ defined by $f(x) = \frac{1}{x}$, $x \in R - \{0\}$. What is the domain and range of this function? Draw the graph of f.

Solution;

The domain is all real numbers except 0 and its range is also all real numbers except 0. The graph of f is given in Fig. 14

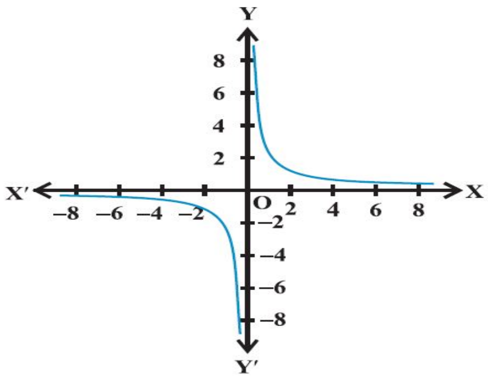

Figure 11: Graph of $f(x) = \frac{1}{x}$

Definition: Surjective (onto)

A function $f : X \rightarrow Y$ is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.

The function f_3 and f_4 in Figure 11 (iii), (iv) are onto and the function f_1 in Figure 11 (i) is not onto as elements e, f in X_2 are not the image of any element in X_1 under f_1.

Definition: Injective (one-one)

A function $f : X \rightarrow Y$ is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e., for every x_1, x_2 $\in X$, $f(x_1) = f(x_2)$ implies $x1 = x2$. Otherwise, f is called many-one.

The function f_1 and f_4 in Figure 11 (i), (iv) are one-one and the function f_2 and f_3 in Figure 11 (i), (iv) are many one.

Definition: Bijective (one-one and onto)

A function $f : X \rightarrow Y$ is said to be one-one and onto (or bijective), if f is both one-one and onto.

The function f_4 in Figure 11 (iv) is one-one and onto.

Example

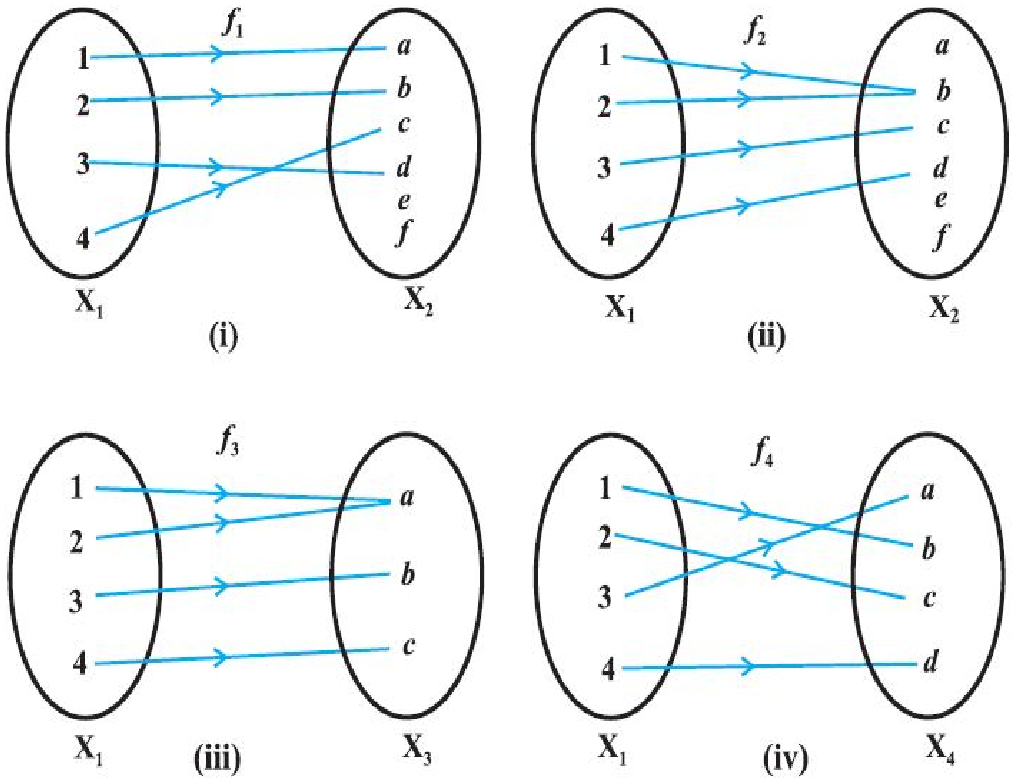

Figure 12: Types of functions

Is the function $f(x) = x^2$ from the set of integers to the set of integers bijection.

Solution

The function f is not onto and one-one because there is no integer x with $x^2 = -1$ and $f(1) = f(-1) = 1$, but $1 \neq -1$.

Hence, f is not bijection.

Definition: Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g, denoted by $g \circ f$, is defined as the function $g \circ f : A \to C$ given by $g \circ f(x) = g(f(x)), \forall x \in A$.

Definition: Invertible function

A function $f : X \to Y$ is defined to be invertible, if there exists a function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called the inverse of f and is denoted by f^{-1}.

Thus, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible.

Example

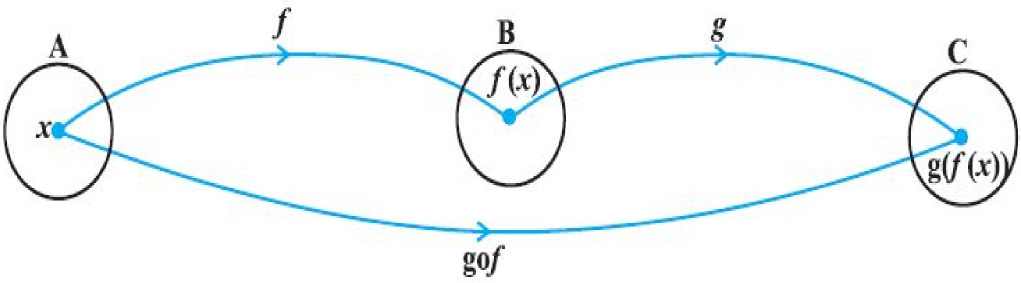

Figure 13: Composition of Functions

Let $f : Z \to Z$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution:

The function f has an inverse because it is a one-to-one correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of Z that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

Theorem

Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then gof is also invertible with $(gof)^{-1} = f^1 og^1$.

Example

Find gof and fog, if $f : R \to R$ and $g : R \to R$ are given by $f(x) = cosx$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

Solution

We have $gof(x) = g(f(x)) = g(cosx) = 3(cosx)^2 = 3cos^2x$. Similarly, $fog(x) = f(g(x)) = f(3x^2) = cos(3x^2)$. Note that $3cos2x \neq cos3x^2, forx = 0$. Hence,$gof \neq fog$.

Example

Show that if $f : A \to B$ and $g : B \to C$ are one-one, then $gof : A \to C$ is also one-one.

Solution

Suppose $gof(x_1) = gof(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$, as g is one-one

$\Rightarrow x_1 = x_2$, as f is one-one

Hence, gof is one-one.

3 Counting

3.1 The basic counting

There are two basic counting principles, the **product rule** and the **sum rule**.

THE PRODUCT RULE

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

EXAMPLE

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:

The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

EXAMPLE

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution:

The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600.

→ An extended version of the product rule is often useful. Suppose that a procedure is carried out by performing the tasks $T_1, T_2, ..., T_m$ in sequence. If each task T_i, $i = 1, 2, \ldots, n$, can be done in ni ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \cdots n_m$ ways to carry out the procedure. This version of the product rule can be proved by mathematical induction from the product rule for two tasks.

EXAMPLE

How many different bit strings of length seven are there?

Solution:

Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

EXAMPLE

Counting Functions How many functions are there from a set with$m$ elements to a set with n elements?

Solution:

Afunction corresponds to a choice of one of the n elements in the codomain for each of themelements in the domain. Hence, by the product rule there are $n \cdot n \cdots n = n^m$ functions from a set with m elements to one with n elements. For example, there are $53 = 125$ different functions from a set with three elements to a set with five elements

EXAMPLE

Counting One-to-One Functions How many one-to-one functions are there from a set with m elements to one with n elements?

Solution:

First note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.

Now let $m \leq n$. Suppose the elements in the domain are $a_1, a_2, ..., a_m$. There are n ways to choose the value of the function at a_1. Because the function is one-to-one, the value of the function at a_2 can be picked in $n - 1$ ways (because the value used for a1 cannot be used again). In general, the value of the function at ak can be chosen in $n - k + 1$ ways. By the product rule, there are $n(n-1)(n-2)\ldots(n-m+1)$ one-to-one functions from a set with m elements to one with n elements.

EXAMPLE

Counting Subsets of a Finite Set Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Solution:

Let S be a finite set. List the elements of S in arbitrary order. We know that there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. Namely, a subset of S is associated with the bit string with a 1 in the ith position if the ith element in the list is in the subset, and a 0 in this position otherwise. By the product rule, there are $2^{|}S|$ bit strings of length $|S|$.

Hence, $|P(S)| = 2|S|$.

THE SUM RULE

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example

Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Solution:

There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick this representative.

$\rightarrow$ We can extend the sum rule to more than two tasks. Suppose that a task can be done in one of n_1 ways, in one of n_2 ways, ... , or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of nj ways, for all pairs i and j with $1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + \ + n_m$. This extended version of the sum rule is often useful in counting problems, as Examples 13 and 14 show. This version of the sum rule can be proved using mathematical induction from the sum rule for two sets.

If $A_1, A_2, ..., A_m$ are pairwise disjoint finite sets then,

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|.$$

3.2 The Pigeonhole Principle

Theorem: The Pigeonhole Principle If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example

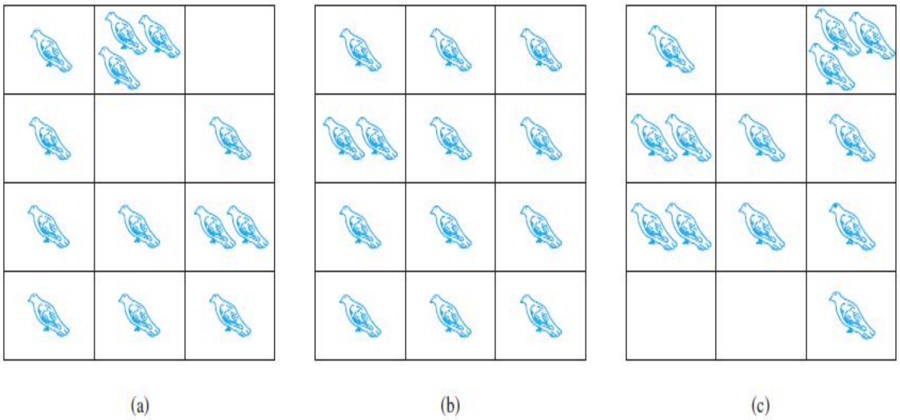

Figure 14: Pigeonhole principle

Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example

In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

THE GENERALIZED PIGEONHOLE PRINCIPLE

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Example

Among 100 people there are at least $\lceil \frac{100}{12} \rceil = 9$ who were born in the same month.

For Example

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solution:

The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil \frac{N}{5} \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

Example

a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen? A standard deck of 52 cards has 13 kinds of cards, with four cards of each of kind, one in each of the four suits, hearts, diamonds, spades, and clubs.

b) How many must be selected to guarantee that at least three hearts are selected?

Solution:

a) Suppose there are four boxes, one for each suit, and as cards are selected they are placed in the box reserved for cards of that suit. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing at least $\lceil \frac{N}{4} \rceil$ cards. Consequently, we know that at least three cards of one suit are selected if $\lceil \frac{N}{4} \rceil \geq 3$. The smallest integer N such that $\lceil \frac{N}{4} \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$, so nine cards suffice. Note that if eight cards are selected, it is possible to have two cards of each suit, so more than eight cards are needed. Consequently, nine cards must be selected to guarantee that at least three cards of one suit are chosen. One good way to think about this is to note that after the eighth card is chosen, there is no way to avoid having a third card of some suit.

b) We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three hearts.

3.3 Permutation and Combinations

Definition: Permutation A permutation of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an

r-permutation.

Theorem

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are,

$P(n, r) =^n P_r = n(n-1)(n-2)\cdots(n-r+1)$

r-permutations of a set with n distinct elements.

COROLLARY

If n and r are integers with $0 \leq r \leq n$, then ${}^nP_r = \frac{n!}{(n-r)!}$.

Example

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is ${}^{100}P_3 = 100 \cdot 99 \cdot 98 = 970, 200$.

Example

How many permutations of the letters ABCDEFGH contain the string ABC ?

Solution:

Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G, and H. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters ABCDEFGH in which ABC occurs as a block.

Definition: Combinations

An r-combination of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$. Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ or ${}^nC_r$ and is called a binomial coefficient.

Theorem

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals ${}^nC_r = \frac{n!}{r!(n.r)!}$

COROLLARY

Let n and r be nonnegative integers with $r \leq n$. Then ${}^nC_r =^n C_{n-r}$.

Example

How many bit strings of length n contain exactly r $1s$?

Solution:

The positions of r $1s$ in a bit string of length n form an r-combination of the set $\{1, 2, 3, \ldots, n\}$. Hence, there are ${}^nC_r$ bit strings of length n that contain exactly r $1s$.

Example

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution:

The number of ways to select the committee is ${}^9C_3 \cdot^{11} C_4 = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27,720.$

3.4 Binomial Coefficients

THE BINOMIAL THEOREM

Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} {}^nC_j x^{n-j} y^j$$
$$= {}^nC_0 x^n + {}^nC_1 x^{n-1} y + \cdots + {}^nC_{n-1} x y^{n-1} + {}^nC_n y^n.$$

Example

What is the expansion of $(x+y)^4$?

Solution:

$$(x+y)^4 = \sum_{j=0}^{4} {}^nC_j x^{4-j} y^j$$
$$= {}^4C_0 x^4 + {}^4C_1 x^3 y + {}^4C_2 x^2 y^2 + {}^4C_3 x^1 y^3 + {}^4C_4 y^4$$
$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Example

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^4$?

Solution:

First, note that this expression equals $(2x+(-3y))^{25}$. By the binomial theorem, we have

$$(2x+(-3y))^{25} = \sum_{j=0}^{25} {}^{25}C_j 2x^{25-j} - 3y^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$, namely,

$${}^{25}C_{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13}$$

.

PASCALS IDENTITY

Let n and k be positive integers with $n \geq k$. Then

$${}^{n+k}C_k = {}^n C_{k-1} + {}^n C_k.$$

Pascals Triangle

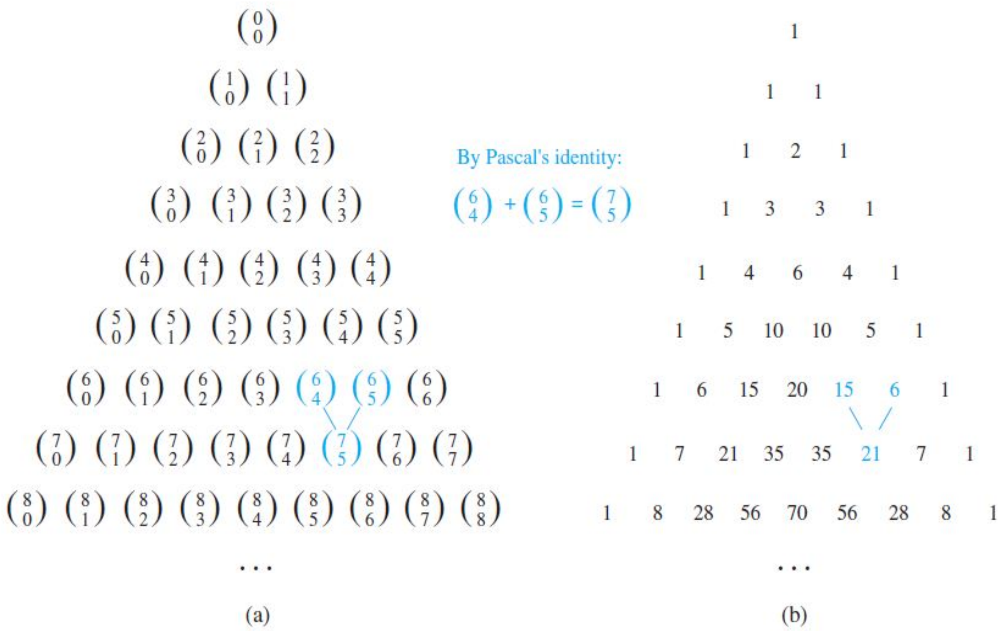

Figure 15: Pascals Triangle

3.5 Generalized Permutations and Combinations

Theorem:Permutations with Repetition

The number of r-permutations of a set of n objects with repetition allowed is n^r.

Example

How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution:

By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly, we see that there are 26^r strings of uppercase English letters of length r.

Theorem: Combinations with Repetition

There are ${}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$ r-combinations from a set with n elements when repetition of elements is allowed.

Example

Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

Solution:

The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. From Theorem 2 this equals ${}^{4+6-1}C_6 = {}^9C_6$. Because ${}^9C_6 = {}^9C_3 = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$, there are 84 different ways to choose the six cookies.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Figure 16: Combinations and Permutations With and Without Repetition

Theorem: Permutations with Indistinguishable Objects

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type $2, \ldots,$ and n_k indistinguishable objects of type k, is

$\frac{n!}{n_1!n_2!n_k!}$

Theorem: Distributing Objects into Boxes

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals

$\frac{n!}{n_1!n_2!n_k!}$

3.6 Generating Permutations and Combinations

Generating Permutations

Any set with n elements can be placed in one to one correspondence with the set $\{1, 2, 3, \ldots, n\}$. We can list the permutations of any set of n elements by generating the permutations of the n smallest positive integers and then replacing these integers with the corresponding elements. Many different algorithms have been developed to generate the $n!$ permutations of this set. We will describe one of these that is based on the lexicographic (or dictionary) ordering of the set of permutations of $\{1, 2, 3, \ldots, n\}$. In this ordering, the permutation $a_1a_2 \cdots a_n$ precedes the permutation of $b_1b_2 \cdots b_n$, if for some k, with $1 \leq k \leq n$, $a_1 = b_1, a_2 = b_2, \ldots a_{k-1} = b_{k-1}$, and $a_k < b_k$. In other words, a permutation of the set of the n smallest positive integers precedes (in lexicographic order) a second permutation if the number in this permutation in the first position where the two permutations disagree is smaller than the number in that position in the second permutation.

Example

What is the next permutation in lexicographic order after 362541?

Solution:

The last pair of integers $a_j and a_{j+1}$ where $a_j < a_{j+1}$ is $a3 = 2$ and $a_4 = 5$. The least integer to the right of 2 that is greater than 2 in the permutation is $a_5 = 4$. Hence, 4 is placed in the third position. Then the integers 2, 5, and 1 are placed in order in the last three positions, giving 125 as the last three positions of the permutation. Hence, the next permutation is 364125.

Generating Combinations

How can we generate all the combinations of the elements of a finite set? Because a combination is just a subset, we can use the correspondence between subsets of $\{a_1, a_2, \ldots, a_n\}$ and bit strings of length n.

Example

Find the next bit string after 10 0010 0111.

Solution:

The first bit from the right that is not a 1 is the fourth bit from the right. Change this bit to a 1 and change all the following bits to 0s. This produces the next larger bit string, 10 0010 1000.