Discrete Mathematics-3140708

Unit-2: Propositional and Predicate Logic

March 20, 2020

Ashok Patel

Assistant Professor

HGCE

Gautam Patel

Assistant Professor

HGCE

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

Contents

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

1 Propositional Logic

1.1 Logic

Definition: Logic

Logic is analysis of language, which consists of signs. Logic is a set of rules or axioms which we can use to draw valid conclusions.

1.2 Propositions

Definition: Propositions (Statements)

Statements are kind of sentence which we have to use to convey our thoughts to others. A sentence is a **statement** or **proposition** if it is possible to say whether what is conveyed by the sentence is true or false.

statements denoted by letters p, q, r, ...

Declarative sentences are said to be propositions.

EXAMPLE

All the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1 and 3 are true, whereas 2 and 4 are false.

Definition: Open statements

An open statement is a sentence that contains one or more variables such that when certain values are substituted for the variables, we get statements.

EXAMPLE

$x + 1 = 2$.

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

1.3 Truth value of a statement

Definition: Truth value of a statement

Statement has a definite truth value which is either true or false. True values are denoted by T and false values are denoted by F.

1.4 Truth table

Definition: Truth tables

A table giving all possible truth values of a statement is called truth table.

Every statement must be either true or false but not both.

If there exists two or more statements, they can be combined to produce a new statement.

These statements are called **compound statements**.

1.5 Connectives

Definition: Conjunction

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition p and q. The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

EXAMPLE

Find the conjunction of the propositions p and q where p is the proposition Rebecca's PC has more than 16 GB free hard disk space and q is the proposition The processor in Rebeccas PC runs faster than 1 GHz.

Solution:

The conjunction of these propositions, $p \wedge q$, is the proposition "Rebeccafs PC has more than 16 GB free hard disk space, and the processor in Rebeccafs PC runs faster than 1 GHz." This conjunction can be expressed more simply as "Rebeccafs PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz." For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

Definition: Disjunction

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

The Truth table for Conjunction and Disjunction.			
p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

EXAMPLE

What is the disjunction of the propositions p and p is the proposition Rebecca's PC has more than 16 GB free hard disk space and q is the proposition The processor in Rebeccas PC runs faster than 1 GHz.

Solution:

The disjunction of p and q, $p \vee q$, is the proposition "Rebecca's PC has at least 16 GB free hard disk space, or the processor in Rebeccas PC runs faster than 1 GHz."

This proposition is true when Rebeccas PC has at least 16 GB free hard disk space, when the PCs processor runs faster than 1 GHz, and when both conditions are true. It is false when both of these conditions are false, that is, when Rebeccas PC has less than 16 GB free hard disk space and the processor in her PC runs at 1 GHz or slower.

Definition: Exclusive

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Definition: Conditional Statements

Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

The Truth Table for the Exclusive and the conditional statements.			
p	q	$p \oplus q$	$p \to q$
T	T	F	T
T	F	T	F
F	T	T	T
F	F	F	T

EXAMPLE

Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement $p \to q$ as a statement in English.

Solution:

From the definition of conditional statements, we see that when p is the statement "Maria learns discrete mathematics" and q is the statement "Maria will find a good job," $p \to q$ represents the statement

"If Maria learns discrete mathematics, then she will find a good job."

There are many other ways to express this conditional statement in English. Among the most natural of these are:

"Maria will find a good job when she learns discrete mathematics."

"For Maria to get a good job, it is sufficient for her to learn discrete mathematics."

and

"Maria will find a good job unless she does not learn discrete mathematics."

Definition: Biconditional Statements

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

The Truth Table for the Biconditional $p \leftrightarrow q$		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

EXAMPLE

Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Find $p \leftrightarrow q$ statement

Solution:

"You can take the flight if and only if you buy a ticket."

This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket but you cannot take the flight (such as when the airline bumps you).

Negation ($\sim$)

Let p be any statement then negation of p is denoted by $\sim p$.

Truth Tables of Compound Propositions

Construct the truth table of the compound proposition $(p \vee \sim q) \rightarrow (p \wedge q)$.

The Truth Table for the Exclusive and the conditional statements.

p	q	$\sim q$	$p \vee \sim q$	$p \wedge q$	$p \vee \sim q \rightarrow p \wedge q$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Contrapositive of an implication The implication $\sim q \rightarrow \sim p$ is called the contrapositive of an ipmplication $p \rightarrow q$.

The Truth Table for $p \rightarrow q$ and $\sim q \rightarrow \sim p$.

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

1.6 Logic and Bit Operations

Logic and Bit Operations

A **bit** is a symbol with two possible values, namely, 0 (zero) and 1 (one). This meaning of the word bit comes from binary digit, because zeros and ones are the digits used in binary representations of numbers. A bit can be used to represent a truth value, because there are two truth values, namely, true and false.That is, 1 represents T(true), 0 represents F (false).A variable is called a **Boolean variable** if its value is either true or false. Consequently, a Boolean variable can be represented using a bit.

Computer **bit operations** correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators $\wedge$, $\vee$, and $\oplus$, the tables shown in following Table for the corresponding bit operations are obtained.We will also use the notation **OR**, **AND**, and **XOR** for the operators $\wedge$, $\vee$, and $\oplus$, as is done in various programming languages.

Table for the Bit Operators **OR**, **AND**, and **XOR**.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Definition: Bit string and length of string

A **bit string** is a sequence of zero or more bits. The **length** of this string is the number of bits in the string.

For example

101010011 is a bit string of length nine.

Example

Find the bitwise OR, bitwise AND, and bitwise XOR of the bit strings 01 1011 0110 and 11 0001 1101. (Here, and throughout this book, bit strings will be split into blocks of four bits to make them easier to read.)

Solution:

The bitwise OR, bitwise AND, and bitwise XOR of these strings are obtained by taking the OR, AND, and XOR of the corresponding bits, respectively. This gives us

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

01 1011 0110

11 0001 1101

---

11 1011 1111 bitwise **OR**

01 0001 0100 bitwise **AND**

10 1010 1011 bitwise **XOR**

1.7 Tautology

Definition: Tautology

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

1.8 Logical Equivalences

Definition: Logical Equivalences

The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Logical Equivalent using truth table

Example

Prove that $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$.

Solution:

The Truth Table for $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$.

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$	$\sim p \wedge \sim q$
T	T	F	T	F	F
T	F	F	T	F	F
F	T	T	T	T	T
F	F	T	F	F	F

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

From table, truth values of $(p \vee q) \wedge \sim p$ and $\sim p \wedge q$ are same for each choice of p and q. Hence $(p \vee q) \wedge \sim p$ is equivalent to $\sim p \wedge q$.

1.9 Duality Law

Two formulas A and A^* are said to be the duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are called dual of each other.

Logical Equivalences.

Equivalence	**Name**
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \vee p \equiv p$	Idempotent laws
$\sim (\sim p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$(p \vee q) \wedge r \equiv (p \vee q) \wedge (p \vee r)$ $(p \wedge q) \vee r \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\sim (p \wedge q) \equiv \sim p \vee \sim q$ $\sim (p \vee q) \equiv \sim p \wedge \sim q$	De Morgans laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \sim p \equiv T$ $p \wedge \sim p \equiv F$	Negation laws

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

Example: State and prone De Morgans laws

Solution:

The De Morgans laws are stated as,

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

The Truth Table for $\sim (p \wedge q) \equiv \sim p \vee \sim q$						
p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

The Truth Table for $\sim (p \vee q) \equiv \sim p \wedge \sim q$						
p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Normal forms

In logic, with the help of truth table we can compare if two statements are equivalent. But when more statements or propositions are involved, then this method is not practical. The standard methods are called **Normal forms** or **canonical forms**.

Disjunctive normal form (DNF)

Disjunctive normal form is a disjunction ($\vee$) of fundamental conjunctions ($\wedge$). Hence disjunction of conjunctions are joining fundamental conjunction by $\vee$.

For example

1. $(p \wedge q) \vee p \vee (q \wedge \sim p)$
2. $(p \wedge q) \vee r$
3. $(p \wedge q) \vee (r \wedge q) \vee (\sim r \wedge p)$

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

Conjunctive normal form (CNF)

Conjunctive normal form is a conjunction ($\wedge$) of fundamental disjunctions ($\vee$).

Hence conjunction of disjunctions are joining fundamental disjunction by $\wedge$.

For example

1. $(p \vee q) \wedge (q \vee r) \wedge (\sim p \vee \sim r)$
2. $p \vee (\sim q \vee \sim r)$

2 Predicate Logic

2.1 Predicate Logic

Definition: Predicate

A propositional function or a predicate in one or more variables is a sentence that becomes a proposition, when variables are given definite values from the set of values it can take.

For example

1. $x > 5$ is a one place predicate and it is denoted by $p(x)$.
2. $x + y + z = 10$ is three place predicate and it is denoted by $p(x, y, z)$

PRECONDITIONS AND POSTCONDITIONS

Predicates are also used to establish the correctness of computer programs, that is, to show that computer programs always produce the desired output when given valid input. (Note that unless the correctness of a computer program is established, no amount of testing can show that it produces the desired output for all input values, unless every input value is tested.) The statements that describe valid input are known as **preconditions** and the conditions that the output should satisfy when the program has run are known as **postconditions**.

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

2.2 Quantifiers

A second method of binding individual variables in a predicate is by quantification of the variable.

2.3 Universal Quantifiers

Universal Quantifiers

The universal quantification of P(x) is the statement P(x) for all values of x in the domain.

The notation $\forall\ xP(x)$ denotes the universal quantification of P(x). Here $\forall$ is called the universal quantifier.We read $\forall\ xP(x)$ as for all $xP(x)$) or for every $xP(x)$. An element for which P(x) is false is called a counterexample of $\forall\ xP(x)$.

For example

Let $P(x)$ be the statement "$x + 1 > 1$". What is the truth value of the quantification $\forall$ $xP(x)$,

where the domain consists of all real numbers?

Solution:

Because $P(x)$ is true for all real numbers x, the quantification

$\forall\ xP(x)$ is true.

2.4 Universal of Discourse

Universal of Discourse

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the domain of discourse (or the universe of discourse), often just referred to as the domain. Such a statement is expressed using universal quantification.

Note that the domain specifies the possible values of the variable x.

2.5 Existential Quantifiers

Existential Quantifiers

The existential quantification of $P(x)$ is the proposition There exists an element x in the domain such that $P(x)$. We use the notation $\exists\ xP(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

For example

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists$ $xP(x)$,

where the domain consists of all real numbers?

Solution:

Because "$x > 3$." is sometimes true. for instance, when $x = 4$.the existential quantification of $P(x)$, which is $\exists\ xP(x)$, is true.

Quantifiers.		
Statement	When True?	When False?
$\forall\ xP(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists\ xP(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Negating Quantified Expressions

For instance, consider the negation of the statement

"Every student in your class has taken a course in calculus."

This statement is a universal quantification, namely,

$\forall\ xP(x)$, where $P(x)$ is the statement "x has taken a course in calculus" and the domain consists of the students in your class. The negation of this statement is "It is not the case that every student in your class has taken a course in calculus." This is equivalent to "There is a student in your class who has not taken a course in calculus." And this is simply the existential quantification of the negation of the original propositional function, namely,

$\exists\ xP(x)$.

This example illustrates the following logical equivalence:

$\sim \forall\ xP(x) \equiv \sim \exists\ xP(x)$.

Ashok Patel Gautam Patel Ashok Patel Gautam Patel

2.6 Free and Bound variables

When a quantifier is used in the variable x or when we assign a value to this variable, then the occurrence of the variable is **bound**. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.

2.7 Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Ashok Patel Gautam Patel Ashok Patel Gautam Patel