# Discrete Mathematics-3140708

# Unit-3: Relations and Partial ordering

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**Ashok Patel**
Assistant Professor
HGCE

**Gautam Patel**
Assistant Professor
HGCE

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# 1 Relations

A relation $R$ from a non-empty set $A$ to a non-empty set $B$ is a subset of the cartesian product $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the **image** of the first element.

The set of all first elements of the ordered pairs in a relation $R$ from a set $A$ to a set $B$ is called the **domain** of the relation $R$.

The set of all second elements in a relation $R$ from a set $A$ to a set $B$ is called the range of the relation $R$. The whole set $B$ is called the **codomain** of the relation $R$.

Note that **range** $\subset$ **codomain**.

**Relations on a Set**

A relation on a set $A$ is a relation from $A$ to $A$.In other words, a relation on a set $A$ is a subset of $A \times A$.

**Example**

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) |\, a\, divides\, b\}$?

**Solution:**

Because $(a, b)$ is in $R$ if and only if a and b are positive integers not exceeding 4 such that $a$ divides $b$, we see that

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

## 1.1 Properties of Relations

**Definition: Reflexive Relation**

A relation $R$ on a set $A$ is called reflexive if $(a, a) \in R$ for every element $a \in A$.

**Example**

If $A$ is a set of all straight lines in 2-D plane and $R$ is a relation.

$R = \{(a, b)$ : a is parallel to $b\}$

them $R$ ia reflexive relation as every straight line is parallel to it self.

**Definition: Transitive Closure**

The transitive closure of a relation $R$ is the smallest transitive relation containing $R$.

Transitive closure of $R$ is denoted by $R^*$.

**Definition: Symmetric Relations**

A relationR on a setAis called symmetric if $(b,a) \in R$ whenever $(a,b) \in R$, for all $a, b \in A$.

**Definition: Antisymmetric Relations**

A relation R on a set A such that for all $a, b \in A$, if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called antisymmetric.

**Definition: Transitive Relations**

A relation $R$ on a set $A$ is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a, b, c \in A$.

**Definition: Identity Relation in a set**

Let $A$ be a set. The relation $I_A$ defined by, $I_A = \{(a,b) : a \in A, b \in A, a = b\}$ is called the identity relation. In other words the identity relation in a set $A$ is the set of ordered pair $(a,b)$ of $A \times A$ for which $a = b$.

**Definition: Universal Relation in a set**

Let $A$ be a set and $R$ be any relation on a set $A$, then $R \subseteq A \times A$.

If $R = A \times A$, then the relation $R$ is known as universal relation.

**Definition: Void or Empty Relation**

A relation $R$ a set $A$ is called void relation. If no element of set $A$ is related to any element of $A$. Hence $R = \phi$

**Example**

Let $A$ be the set of all students of a boys school. Show that the relation $R$ in $A$ given by $R = \{(a,b) : a$ is sister of $b\}$ is the empty relation and $R = \{(a,b) :$ the difference between heights of $a$ and $b$ is less than 3 meters$\}$ is the universal relation.

**Solution:**

Since the school is boys school, no student of the school can be sister of any student of the school. Hence, $R = \phi$, showing that $R$ is the empty relation. It is also obvious that the difference between heights of any two students of the school has to be less than 3 meters. This shows that $R = A \times A$ is the universal relation.

**Example**

Consider the following relations on $\{1, 2, 3, 4\}$:

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$,

$R_2 = \{(1,1),(1,2),(2,1)\}$,

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

Which of these relations are reflexive, transitive, symmetric and antisymmetric?

**Solution:**

The relations $R_3$ and $R_5$ are reflexive because they both contain all pairs of the form (a, a), namely, (1, 1), (2, 2), (3, 3), and (4, 4). The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, $R_1$, $R_2$, $R_4$, and $R_6$ are not reflexive because (3, 3) is not in any of these relations.The relations $R_2$ and $R_3$ are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. For $R_2$, the only thing to check is that both (2, 1) and (1, 2) are in the relation. For $R_3$, it is necessary to check that both (1, 2) and (2, 1) belong to the relation, and (1, 4) and (4, 1) belong to the relation. The reader should verify that none of the other relations is symmetric. This is done by finding a pair (a, b) such that it is in the relation but (b, a) is not. $R_4$, $R_5$, and $R_6$ are all antisymmetric. For each of these relations there is no pair of elements a and b with $a = b$ such that both (a, b) and (b, a) belong to the relation. The reader should verify that none of the other relations is antisymmetric. This is done by finding a pair (a, b) with $a = b$ such that (a, b) and (b, a) are both in the relation. $R_4$, $R_5$, and $R_6$ are transitive. For each of these relations, we can show that it is transitive by verifying that if (a, b) and (b, c) belong to this relation, then (a, c) also does. For instance, $R_4$ is transitive, because (3, 2) and (2, 1), (4, 2) and (2, 1), (4, 3) and (3, 1), and (4, 3) and (3, 2) are the only such sets of pairs, and (3, 1), (4, 1), and (4, 2) belong to $R_4$. The reader should verify that $R_5$ and $R_6$ are transitive. $R_1$ is not transitive because (3, 4) and (4, 1) belong to $R_1$, but (3, 1) does not. $R_2$ is not transitive because (2, 1) and (1, 2) belong to $R_2$, but (2, 2) does not. $R_3$ is not transitive because (4, 1) and (1, 2) belong to $R_3$, but (4, 2) does not.

**Example**

Show that the relation $R$ in the set $\{1, 2, 3\}$ given by $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$ is reflexive but neither symmetric nor transitive.

**Solution:**

$R$ is reflexive, since $(1, 1), (2, 2)$ and $(3, 3)$ lie in $R$. Also, $R$ is not symmetric,as $(1, 2) \in R$ but $(2, 1) \notin R$. Similarly, $R$ is not transitive, as $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$.

**Example**

Let $L$ be the set of all lines in a plane and $R$ be the relation in $L$ defined as $R = \{(L_1, L_2) : L_1 \text{ is perpendicular to } L_2\}$. Show that $R$ is symmetric but neither reflexive nor transitive.

**Solution:**

$R$ is not reflexive, as a line $L_1$ can not be perpendicular to itself, i.e., $(L_1, L_1) \notin R$. $R$ is symmetric as $(L_1, L_2) \in R$.

$L_1$ is perpendicular to $L_2$

$L_2$ is perpendicular to $L_1$

$(L_2, L_1) \in R$. $R$ is not transitive. Indeed, if $L_1$ is perpendicular to $L_2$ and $L_2$ is perpendicular to $L_3$, then $L_1$ can never be perpendicular to $L_3$. In fact, $L_1$ is parallel to $L_3$, i.e., $(L_1, L_2) \in R, (L_2, L_3) \in R$ but $(L_1, L_3) \notin R$.

**Definition: Composite Relations**

Let $R$ be a relation from a set $A$ to a set $B$ and $S$ be a relation from a set $B$ to a set $C$. The composite of $R$ and $S$ is the relation consisting of ordered pairs $(a, c)$, where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $R$ and $S$ by $S \circ R$.

**Example**

What is the composite of the relations $R$ and $S$, where $R$ is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and $S$ is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

**Solution:**

$S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs (2, 3) in R and (3, 1) in S produce the ordered pair (2, 1) in S .R. Computing all the ordered pairs in the composite, we find $S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

## 1.2 Combining Relations

**Example**

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$,

$R_1 \cap R_2 = \{(1,1)\}$,

$R_1 - R_2 = \{(2,2),(3,3)\}$,

$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$.

**Example**

Let A and B be the set of all students and the set of all courses at a school, respectively. Suppose that $R_1$ consists of all ordered pairs (a, b), where a is a student who has taken course b, and $R_2$ consists of all ordered pairs (a, b), where a is a student who requires course b to graduate.

What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R1 \oplus R_2$, $R_1 R_2$, and $R_2 R_1$?

**Solution:**

The relation $R_1 \cup R_2$ consists of all ordered pairs (a, b), where a is a student who either has taken course b or needs course b to graduate, and $R_1 \cap R_2$ is the set of all ordered pairs (a, b), where a is a student who has taken course b and needs this course to graduate. Also, $R1 \oplus R_2$ consists of all ordered pairs (a, b), where student a has taken course b but does not need it to graduate or needs course b to graduate but has not taken it. $R_1 - R_2$ is the set of ordered pairs (a, b), where a has taken course b but does not need it to graduate; that is, b is an elective course that a has taken. $R_2 - R_1$ is the set of all ordered pairs (a, b), where b is a course that a needs to graduate but has not taken.

**Example**

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$. Find the powers $R_n$, n = 2, 3, 4, . . . .

Because $R^2 = R \circ R$, we find that $R_2 = \{(1,1),(2,1),(3,1),(4,2)\}$. Furthermore, because $R_3 = R_2 \circ R$, $R_3 = \{(1,1),(2,1),(3,1),(4,1)\}$. Additional computation shows that $R_4$ is the same as $R_3$, so $R_4 = \{(1,1),(2,1),(3,1),(4,1)\}$. It also follows that $R_n = R_3$ for n = 5, 6, 7, . . . .



## 1.3 Representing Relations Using Matrices

A relation between finite sets can be represented using a zer-one matrix. Suppose that $R$ is a relation from $A = \{a_1, a_2, ..., a_m\}$ to $B = \{b_1, b_2, ..., b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$ we use the same ordering for $A$ and $B$.) The relation $R$ can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$,where

$$mij = 1 \ if \ (a_i, b_j) \in R,$$
$$= 0 \ if \ (a_i, b_j) \notin R.$$

In other words, the zero-one matrix representing $R$ has a 1 as its $(i, j)$ entry when $a_i$ is related to $b_j$, and a 0 in this position if $a_i$ is not related to $b_j$.

**Example**

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let $R$ be the relation from $A$ to $B$ containing $(a, b)$ if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing $R$ if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

**Solution:**

Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for $R$ is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1s in $\mathbf{M}_R$ show that the pairs $(2, 1)$, $(3, 1)$, and $(3, 2)$ belong to $R$. The 0s show that no other pairs belong to $R$.

**Example**

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation $R$ represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

**Solution:**

Because Because $R$ consists of those ordered pairs $(a_i, b_j)$ with $m_{ij} = 1$, it follows that $R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$.

The Zero-One Matrix for Reflexive Relation, symmetric and anti-symmetry (see Fig. 1).

**Example**

Suppose that the relation $R$ on a set is represented by the matrix $\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

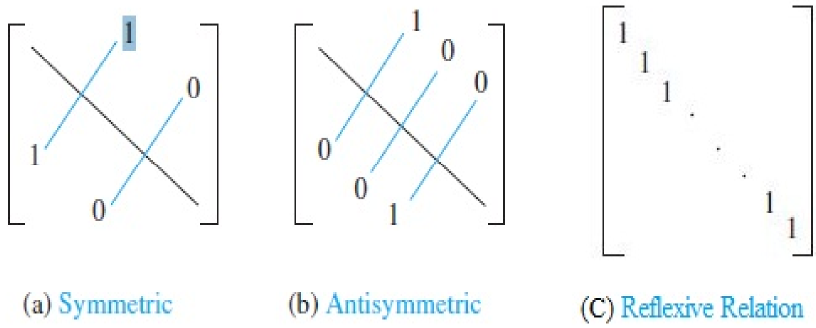


Figure 1: Zero-One Matrix for Reflexive Relation, symmetric and anti-symmetry

Is $R$ reflexive, symmetric, and/or antisymmetric?

**Solution:**

Because all the diagonal elements of this matrix are equal to 1, $R$ is reflexive. Moreover, because $\mathbf{M}_R$ is symmetric, it follows that $R$ is symmetric. It is also easy to see that $R$ is not antisymmetric.

## 1.4 Relation Matrix operations

Suppose that $R_1$ and $R_2$ are relations on a set A represented by the matrices $\mathbf{M}_{R_1}$ and $\mathbf{M}_{R_2}$ , respectively. The matrix representing the union of these relations has a 1 in the positions where either $\mathbf{M}_{R_1}$ or $\mathbf{M}_{R_2}$ has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both $\mathbf{M}_{R_1}$ and $\mathbf{M}_{R_2}$ have a 1. Thus, the matrices representing the union and intersection of these relations are

$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2}$ and $\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}$. **Properties of relation Matrix** Let $R_1$ be a relation from $A$ to $B$ and $R_2$ be a relation from $B$ to $C$. Then the relation matrices satisfy the following properties,

$$\begin{aligned}
\mathbf{M}_{R_1 \cdot R_2} &= \mathbf{M}_{R_1} \cdot \mathbf{M}_{R_2} \\
\mathbf{M}_{R_{-1}} &= \text{Transpose of } \mathbf{M}_R \\
\mathbf{M}_{(R_1 \cdot R_2)^{-1}} &= \mathbf{M}_{R_1^{-1}} \cdot \mathbf{M}_{R_2^{-1}}
\end{aligned}$$

where the product is known as **Boolean product**.

## 1.5 Representing Relations Using Digraphs

Let $A$ be finite set and $R$ be a relation defined on $R$, then $R$ can be represented by graph known as directed graphs or digraphs.

**Definition: Digraphs**

A directed graph, or digraph, consists of a set $V$ of vertices (or nodes) together with a set $E$ of ordered pairs of elements of $V$ called edges (or arcs). The vertex $a$ is called the initial vertex of the edge $(a, b)$, and the vertex $b$ is called the terminal vertex of this edge. An edge of the form $(a, a)$ is represented using an arc from the vertex a back to itself. Such an edge is called a loop.

**For example**

The directed graph with vertices $a$, $b$, $c$, and $d$, and edges $(a, b)$, $(a, d)$, $(b, b)$, $(b, d)$, $(c, a)$, $(c, b)$, and $(d, b)$ is displayed in Fig. 1,

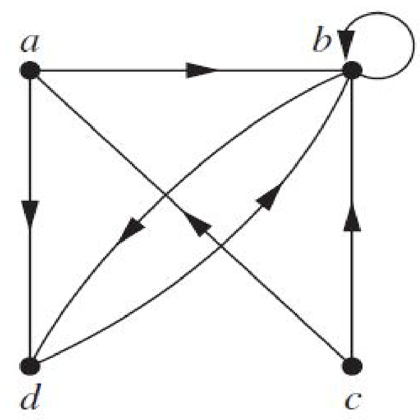


Figure 2: A Directed Graph



## 1.6 Equivalence Relations

**Definition**

A relation on a set $A$ is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements a and b that are related by an equivalence relation are called equivalent. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

**Example**

Let $R$ be the relation on the set of real numbers such that $aRb$ if and only if $a - b$ is an integer.

Is $R$ an equivalence relation?

**Solution:**

Because $a - a = 0$ is an integer for all real numbers $a$, $aRa$ for all real numbers a. Hence, R is reflexive. Now suppose that $aRb$. Then $a - b$ is an integer, so $ba$ is also an integer. Hence, $bRa$. It follows that $R$ is symmetric. If $aRb$ and $bRc$, then $a - b$ and $bc$ are integers. Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, $aRc$. Thus, $R$ is transitive. Consequently, $R$ is an equivalence relation.

**Example: Congruence Modulo $m$**

Let $m$ be an integer with $m > 1$. Show that the relation $R = \{(a, b)| \ a \equiv b \ (\text{mod m})\}$ is an equivalence relation on the set of integers.

**Solution:**

$a \equiv b$ (mod m) if and only if $m$ divides $a - b$. Note that $a - a = 0$ is divisible by $m$, because$0 = 0 \cdot m$. Hence, $a \equiv a$ (mod m), so congruence modulo $m$ is reflexive. Now suppose that $a \equiv b$ (mod m). Then $a - b$ is divisible by $m$, so $a - b = km$, where $k$ is an integer. It follows that $b - a = (-k)m$, so $b \equiv a$ (mod m). Hence, congruence modulo $m$ is symmetric. Next, suppose that $a \equiv b$ (mod m) and $b \equiv c$ (mod m). Then $m$ divides both $a - b$ and $b - c$. Therefore, there are integers $k$ and $l$ with $a - b = km$ and $b - c = lm$. Adding these two equations shows that

$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$

Thus, $a \equiv c$ (mod m). Therefore, congruence modulo $m$ is transitive. It follows that congruence modulo $m$ is an equivalence relation.

## 1.7 Equivalence Classes

Let $R$ be an equivalence relation on a set $A$. The set of all elements that are related to an element a of $A$ is called the equivalence class of $a$. The equivalence class of a with respect to $R$ is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript $R$ and write $[a]$ for this equivalence class.

In other words, if $R$ is an equivalence relation on a set $A$, the equivalence class of the element $a$ is

$[a]_R = s|(a,s) \in R$.

**Example**

What are the equivalence classes of 0 and 1 for congruence modulo 4?

**Solution:**

$[0] = \{..., 8, 4, 0, 4, 8, ...\}$.

$[1] = \{..., 7, 3, 1, 5, 9, ...\}$.

## 1.8 Partition of a Set

Let $S$ be a nonempty finite set. A decomposition of $S$ of the form $S = \bigcup_{i=1}^{k} A_i$ where $A_i \neq \phi$ for all $i = 1, 2, \ldots, k$ is called a covering of $S$. If, in addition, $A_i \cap A_j = \phi$ for $i \neq j$ , then $\{A_1, A_2, \ldots, A_k\}$ is called a partition of $S$ and the sets $A_1, A_2, \ldots, A_k$ are called the blocks of the partition.

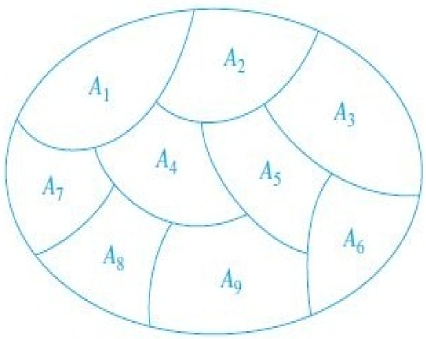


Figure 3: Partition of a Set

**Definition: Compatible Relation**

A relation $R$ on a set $S$ is said to be compatible if it is reflexive and Symmetric.

**Definition: Maximal compatibility block**

A subset $A \subseteq X$ is said to be a maximal compatibility block if any element of $A$ is compatible to all other elements of $A$ and no element of $X - A$ is compatible to all the elements of $A$.

**For Example**

Let $X = \{A_1, A_2, A_3, A_4, A_5\}$ where $A_1 = \{1, 2\}$, $A_2 = \{2, 3, 4\}$, $A_3 = \{3, 4, 5\}$, $A_4 = \{2, 4\}$, $A_5 = \{4, 5\}$ and let $R$ be given by

$R = \{(A_i, A_j)|A_i, A_j \in X$ and $A_i R A_j$ if $A_i$ and $A_j$ contain some common element$\}$.

**Solution:**

From Fig. 4, $\{A_1, A_2, A_4\}$, $\{A_2, A_3, A_4\}$, $\{A_2, A_4, A_5\}$, $\{A_3, A_4, A_5\}$, $\{A_2, A_3, A_4, A_5\}$ are

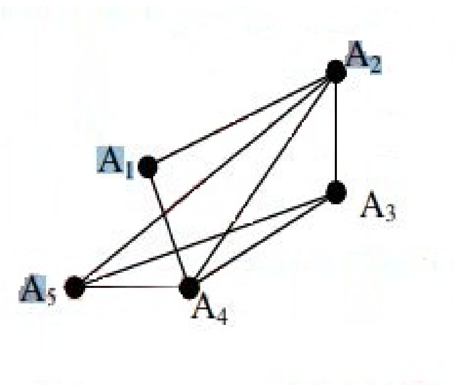


Figure 4: Maximal compatibility block

mutually compatible and the sets are not mutually disjoint.

Clearly, $\{A_1, A_2, A_4\}$ and $\{A_2, A_3, A_4, A_5\}$ are the only maximal compatibility blocks

## 2 Partial Ordering

**Definition: Partial ordering Relations**

A relation $R$ on a set $S$ is called a partial ordering or partial order if it is reflexive, anti-symmetric, and transitive. A set $S$ together with a partial ordering $R$ is called a partially ordered set, or poset, and is denoted by $(S, R)$. Members of $S$ are called elements of the poset.

**Example**

Show that the greater than or equal relation $(\geq)$ is a partial ordering on the set of integers.

**Solution:**

Because $a \geq a$ for every integer $a$, $\geq$ is reflexive. If $a \geq b$ and $b \geq c$, then $a = b$. Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering on the set of integers and $(Z, \geq)$ is a poset.

**Example**

Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set $S$.

**Solution:**

Because $A \subseteq A$ whenever $A$ is a subset of S, $\subseteq$ is reflexive. It is anti-symmetric because

$A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally, $\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset.

**Definition: Comparable posets**

The elements $a$ and $b$ of a poset $(S, \leq)$ are called comparable if either $a \leq b$ or $b \leq a$. When $a$ and $b$ are elements of$S$ such that neither $a \leq b$ nor $b \leq a$, $a$ and $b$ are called incomparable.

**Example**

In the poset $(Z+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

**Solution:**

The integers 3 and 9 are comparable, because $3|9$. The integers 5 and 7 are incomparable, because $5 \not| 7$ and $7 \not| 5$.

**Definition: Linear ordering or Totally ordering**

If $(S, \leq)$ is a poset and every two elements of $S$ are comparable, $S$ is called a totally ordered or linearly ordered set, and $\leq$ is called a total order or a linear order. A totally ordered set is also called a chain.

**Example**

The poset $(Z, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever $a$ and $b$ are integers.

**Example**

The poset $(Z+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7.

**Definition: Well-ordered set**

$(S, \leq)$ is a well-ordered set if it is a poset such that $\leq$ is a total ordering and every nonempty subset of $S$ has a least element.

**Example**

The set of ordered pairs of positive integers, $Z^+ \times Z^+$, with $(a_1, a_2) \leq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2.b_2$ (the lexicographic ordering), is a well-ordered set. The set $Z$, with the usual $\leq$ ordering, is not well-ordered because the set of negative integers, which is a subset of $Z$, has no least element

**Definition: Lexicographic Order**

The lexicographic ordering $\leq$ on $A1 \times A2$ is defined on two posets, $(A_1, \leq_1)$ and $(A_2, \leq_2)$ by specifying that one pair is less than a second pair if the first entry of the first pair is less than (in $A_1$) the first entry of the second pair, or if the first entries are equal, but the second entry of this pair is less than (in $A_2$) the second entry of the second pair. In other words, $(a_1, a_2)$ is less than $(b_1, b_2)$, that is,

$(a_1, a_2) < (b_1, b_2)$, either if $a_1 <_1 b_1$ or if both $a_1 = b_1$ and $a_2 <_1 b_2$.

**Example**

Determine whether $(3, 5) < (4, 8)$, whether $(3, 8) < (4, 5)$, and whether $(4, 9) < (4, 11)$ in the poset $(Z \times Z, \leq)$, where $\leq$ is the lexicographic ordering constructed from the usual $\leq$ relation on Z.

**Solution:**

Because $3 < 4$, it follows that $(3, 5) < (4, 8)$ and that $(3, 8) < (4, 5)$. We have $(4, 9) < (4, 11)$, because the first entries of $(4, 9)$ and $(4, 11)$ are the same but $9 < 11$.

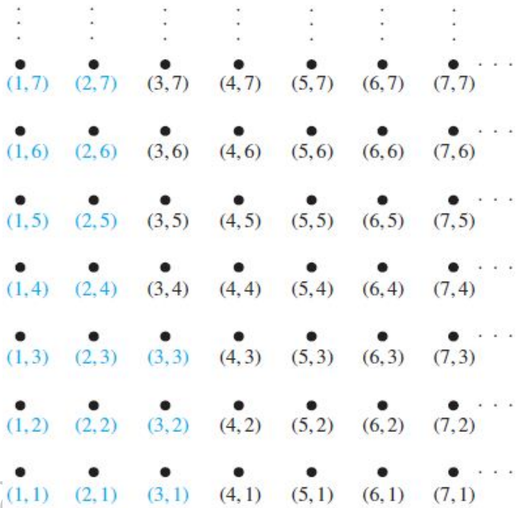


Figure 5: The Ordered Pairs Less Than $(3, 4)$ in Lexicographic Order.

**Hasse Diagrams**

Poset can be represented by digraphs. A simpler way of representing poset is Hasse diagram.

**Method for finding Hasse diagram**

**1**. Omit loops as relation is reflexive on poset.

**2**. All arrows that appear on the edges are omitted.

**3**. Eliminate all edges that are implied by transitive relation.

**4**. An arc pointing upward is drawn from $a$ to $b$ if $a \neq b$ and $aRb$.

**Example**

Constructing the Hasse Diagram for $(\{1, 2, 3, 4\}, \leq)$.

**Solution:**

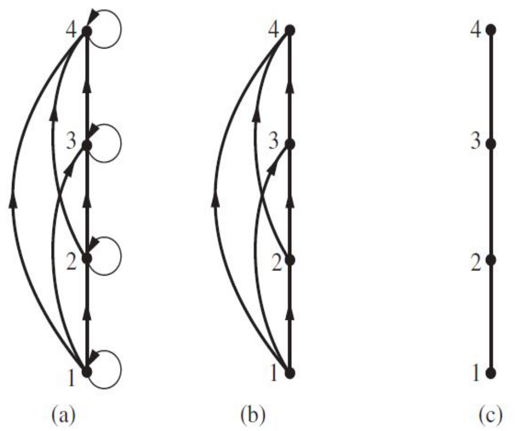


Figure 6: Constructing the Hasse Diagram for $(\{1, 2, 3, 4\}, \leq)$

**Example**

Draw the Hasse diagram representing the partial ordering $\{(a, b)\, a \, \text{dividse} \, b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

**Solution:**

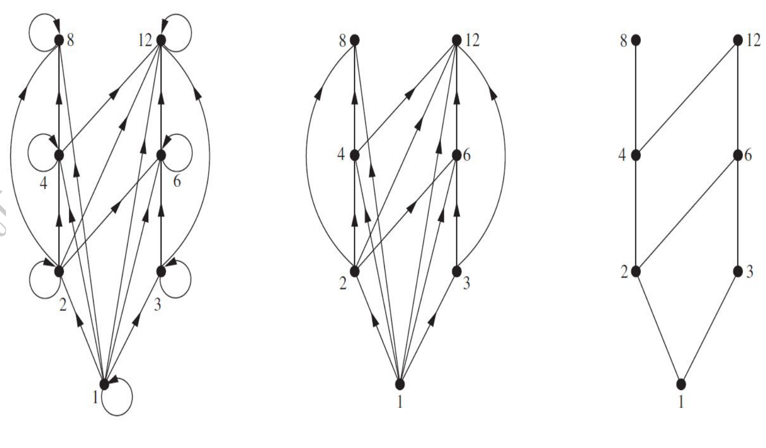


Figure 7: Constructing the Hasse Diagram for $(\{1, 2, 3, 4, 6, 8, 12\}, |)$.



**Definition: Maximal and Minimal Elements**

Elements of posets that have certain extremal properties are important for many applications. An element of a poset is called maximal if it is not less than any element of the poset. That is, a is maximal in the poset $(S, \leq)$ if there is no $b \in S$ such that $a < b$. Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is minimal if there is no element $b \in S$ such that $b < a$. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the **top** and **bottom** elements in the diagram.

**Definition: Least and Greatest Elements**

$a$ is the greatest element of the poset $(S, \leq)$ if $b \leq a$ for all $b \in S$. The greatest element is unique when it exists.

Likewise, an element is called the least element if it is less than all the other elements in the poset. That is, a is the least element of $(S, \leq)$ if $a \leq b$ for all $b \in S$. The least element is unique when it exists.

**Example**

Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal?

**Solution:**

The Hasse diagram in Figure 8 for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element.

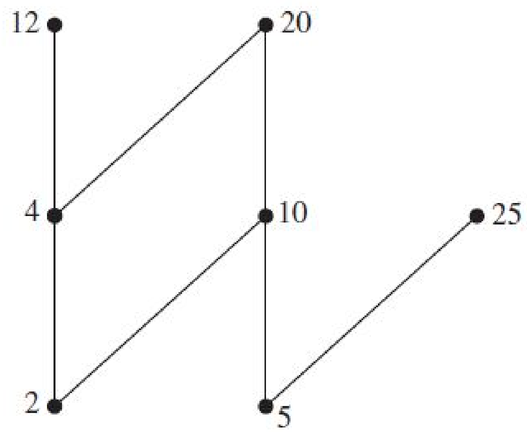


Figure 8: The Hasse Diagram of a Poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$

**Example**

Determine whether the posets represented by each of the Hasse diagrams in Figure 9 have a greatest element and a least element.

**Solution:**

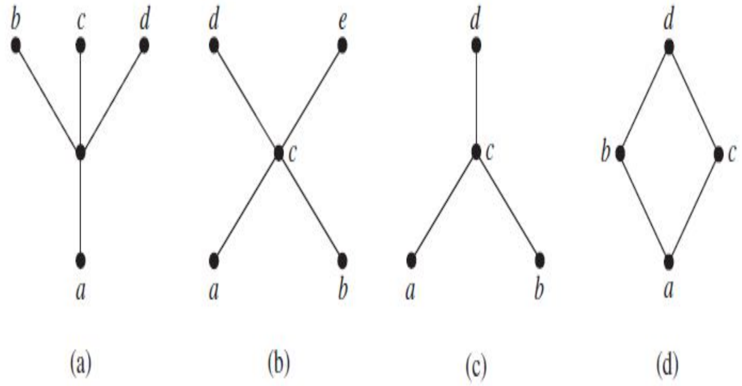


Figure 9: Hasse Diagrams of Four Posets.

The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element. The poset with Hasse diagram (b) has neither a least nor a greatest element. The poset with Hasse diagram (c) has no least element. Its greatest element is d. The poset with Hasse diagram (d) has least element a and greatest element d.

**Definition: Upper bound and Lower bound**

If $u$ is an element of $S$ such that $a \leq u$ for all elements $a \in A$, then $u$ is called an upper bound of $A$. Likewise, there may be an element less than or equal to all the elements in $A$. If $l$ is an element of $S$ such that $l \leq a$ for all elements $a \in A$, then $l$ is called a lower bound of $A$.

**Example**

Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 10.

**Solution:**

The upper bounds of $\{a, b, c\}$ are $e, f, j$, and $h$, and its only lower bound is $a$. There are no upper bounds of $\{j, h\}$, and its lower bounds are $a, b, c, d, e$, and $f$ . The upper bounds of $\{a, c, d, f\}$ are $f, h, andj$ , and its lower bound is $a$.

**Definition: Least upper bound and Greatest lower bound**

The element $x$ is called the least upper bound (Supremum) of the subset $A$ if $x$ is an upper bound that is less than every other upper bound of $A$. Because there is only one such element, if it exists, it makes sense to call this element the least upper bound . That is, $x$ is the least upper bound of $A$ if $a \leq x$ whenever $a \in A$, and $x \leq z$ whenever $z$ is an upper bound of $A$. Similarly, the element $y$ is called the greatest lower bound (Infimum)

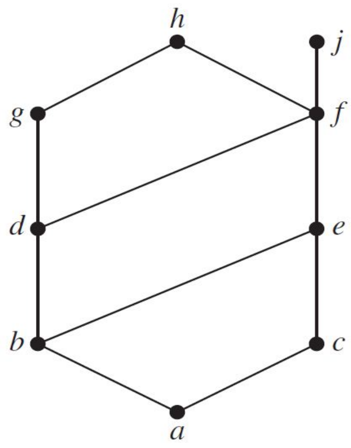


Figure 10: Hasse Diagrams of a Poset.

of $A$ if $y$ is a lower bound of $A$ and $z \leq y$ whenever $z$ is a lower bound of $A$. The greatest lower bound of $A$ is unique if it exists. The greatest lower bound and least upper bound of a subset A are denoted by $glb(A)$ and $lub(A)$, respectively.

**Example**

Find the lower and upper bounds of $\{b, d, g\}$ if they exist, in the poset shown in Figure 10.

**Solution:**

The upper bounds of $\{b, d, g\}$ are $g$ and h. Because $g < h$, $g$ is the least upper bound. The lower bounds of $\{b, d, g\}$ are $a$ and $b$. Because $a < b$, $b$ is the greatest lower bound.

**Definition: Lattices**

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice. Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra. Lattice is denoted by the notation $(A, *, \oplus)$ for poset $(A, \leq)$.

**Example**

Is the poset $(Z^+, |)$ a lattice?

**Solution:**

Let a and b be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this poset is a lattice.

**Lattice operators**

In lattice lub of $a$ and $b$ is denoted by $a * b$ and it is known as $a$ join $b$. Similarly glb of $a$ and $b$ is denoted by $a \oplus b$ and is known as $a$ meet $b$.

Other symbol such as $\vee$ and $\wedge$ or $\cdot$ and $+$ or $\cap$ and $\cup$.

**Types of Lattices**

**Distributive Lattice**

A lattice $(A, \leq)$ is called distributive lattice if for any elements $a, b, c \in A$.

(i) $a \oplus (b * c) = (a \oplus b) \oplus (a \oplus c)$.

(ii) $a * (b \oplus c) = (a \oplus b) \oplus (a * c)$.

**Complete Lattice**

A lattice is called complete if each of its non-empty subsets has a lub and a glb.

**For example**

Every finite Lattice must be complete.

The least and the greatest elements of a Lattice, if they exist, are called the bounds (units) of the Lattice and are denoted by 0 and 1 respectively. This type of Lattice is known as bounded Lattice.

**Complemented Lattice**

A bounded lattice $(A, *, \oplus)$ is said to be a complemented lattice if every elements of $A$ has at least one complement.

**Modular Lattice**

A lattice is said to be modular if

$a \leq c \Rightarrow a \oplus (b * c) = (a \oplus b) * c$.

**Definition: Boolean Algebra**

A boolean algebra is a complemented, distributive lattice.

# 3 Recurrence Relations

A recurrence relation for the sequence relation for the sequence $a_0, a_1, \ldots$ is an equation that relates $a_n$ to certain of its predecessors $a_0, a_1, \ldots, a_{n-1}$.

**Example**

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 2$. What are $a_1, a_2$, and $a_3$?

**Solution:**

We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$.

**Example**

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are $a_2$ and $a_3$?

**Solution:**

We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$ and $a_3 = a_2 - a_1 = 25 = 3$.We can find $a_4$, $a_5$, and each successive term in a similar way.

**Fibonacci numbers**

The Fibonacci sequence, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$

**Example**

Find the Fibonacci numbers $f_2, f_3, f_4, f_5$, and $f_6$.

**Solution:**

$f_2 = f_1 + f_0 = 1 + 0 = 1,$

$f_3 = f_2 + f_1 = 1 + 1 = 2,$

$f_4 = f_3 + f_2 = 2 + 1 = 3,$

$f_5 = f_4 + f_3 = 3 + 2 = 5,$

$f_6 = f_5 + f_4 = 5 + 3 = 8.$

**Example**

Suppose that $\{a_n\}$ is the sequence of integers defined by $a_n = n!$, the value of the factorial function at the integer $n$, where $n = 1, 2, 3, \ldots$ Because $n! = n((n-1)(n-2)\ldots 2 \cdot 1) = n(n-1)! = na_{n-1}$, we see that the sequence of factorials satisfies the recurrence relation $a_n = na_{n-1}$, together with the initial condition $a_1 = 1$.

We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula** , for the terms of the sequence

**Example**

Solve the recurrence relation and initial condition that satisfies the recurrence relation $a_n = a_{n1} + 3$ for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 2$

. **Solution:**

We can successively apply the recurrence relation, starting with the initial condition

$a_1 = 2$, and working upward until we reach an to deduce a closed formula for the sequence.We see that

$$
\begin{aligned}
a_2 =& 2 + 3 \\
a_3 =& (2 + 3) + 3 = 2 + 3 \cdot 2 \\
a_4 =& (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3 \\
&. \\
&. \\
&. \\
a_n =& a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1).
\end{aligned}
$$

We can also successively apply the recurrence relation in Example 5, starting with the term an and working downward until we reach the initial condition $a_1 = 2$ to deduce this same formula. The steps are

$$
\begin{aligned}
a_n =& a_{n-1} + 3 \\
=& (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
=& (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&. \\
&. \\
&. \\
=& a_2 + 3(n - 2) = (a_1 + 3) + 3(n - 2) = 2 + 3(n - 1).
\end{aligned}
$$

At each iteration of the recurrence relation, we obtain the next term in the sequence by adding 3 to the previous term. We obtain the nth term after $n - 1$ iterations of the recurrence relation. Hence, we have added 3(n  1) to the initial term $a_0 = 2$ to obtain an. This gives us the closed formula $a_n = 2 + 3(n - 1)$. Note that this sequence is an arithmetic progression.

The technique used in above Example is called **iteration**.We have iterated, or repeatedly used, the recurrence relation. The first approach is called **forward substitution** we found successive terms beginning with the initial condition and ending with $a_n$. The second approach is called **backward substitution**, because we began with $a_n$ and iterated to express it in terms of falling terms of the sequence until we found it in terms of $a_1$.

**Example**

Compound Interest Suppose that a person deposits $10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

**Solution:**

To solve this problem, let $P_n$ denote the amount in the account after $n$ years. Because the amount in the account after $n$ years equals the amount in the account after $n-1$ years plus interest for the $n$th year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$P_n = P_{n-1} + (0.11)P_{n-1} = (1.11)P_{n-1}$. The initial condition is $P_0 = 10,000$. We can use an iterative approach to find a formula for $P_n$. Note that

$$\begin{aligned}
P_1 =& (1.11)P_0 \\
P_2 =& (1.11)P_1 = (1.11)^2 P_0 \\
P_3 =& (1.11)P2 = (1.11)^3 P_0 \\
&. \\
&. \\
&. \\
P_n =& (1.11)P_{n-1} = (1.11)^n P_0.
\end{aligned}$$

When we insert the initial condition $P_0 = 10,000$, the formula $P_n = (1.11)^n 10,000$ is obtained. Inserting n = 30 into the formula $P_n = (1.11)^n 10,000$ shows that after 30 years the account contains $P_{30} = (1.11)^{30} \cdot 10,000 = \$228,922.97$.

**Example**

How can we produce the terms of a sequence if the first 10 terms are $1, 3, 4, 7, 11, 18, 29, 47, 76, 123$?

**Solution:**

Observe that each successive term of this sequence, starting with the third term, is the sum of the two previous terms. That is, $4 = 3 + 1$, $7 = 4 + 3$, $11 = 7 + 4$, and so on. Consequently, if $L_n$ is the nth term of this sequence, we guess that the sequence is determined by the recurrence relation $L_n = L_{n1} + L_{n2}$ with initial conditions $L_1 = 1$ and $L_2 = 3$ (the same recurrence relation as the Fibonacci sequence, but with different initial conditions). This sequence is known as the Lucas sequence.

| *nth Term* | *First 10 Terms* |
|---|---|
| $n^2$ | 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ... |
| $n^3$ | 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ... |
| $n^4$ | 1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ... |
| $2^n$ | 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ... |
| $3^n$ | 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ... |
| $n!$ | 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ... |
| $f_n$ | 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... |

Figure 11: Some Useful Sequences..