Probability and Statistics - 3130006

Unit-1: Basic Probability

July 24, 2020

Contents

1 Probability

1.1 Important terms and concepts

Definition 1 (Random experiment) *An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a random experiment.*

e. g. tossing a coin.

Definition 2 (Outcome) *The result of a random experiment is called an outcome.*

For example, Gives two possible outcomes head or tail when perform a random experiment 'coin is tossed'.

Definition 3 (Exhaustive event) *The total number of possible outcome of a random experiment is called an exhaustive event.*

For Example, In tossing of a coin there are two exhaustive events, head and tail.

Definition 4 (Independent events) *If the occurrence of an event does not have any effect on the occurrence of other events then it is called independent events.*

For example, the event of getting a head in first toss is independent of getting a head in the second, third, and subsequent tosses in tossing a coin.

Definition 5 (Sample Spaces) *The set of all possible outcomes of a random experiment is called the sample space of the experiment. The sample space is denoted as S.*

A sample space is discrete if it consists of a finite or countable infinite set of outcomes. A sample space is continuous if it contains an interval (either finite or infinite) of real numbers.

Definition 6 (Event) *An event is a subset of the sample space of a random experiment*

Definition 7 (mutually exclusive events) *Two events, denoted as E_1 and E_2, such that $E_1 \cap E_2 = \phi$ are said to be mutually exclusive.*

1.2 Probability definition

Definition 8 (Probability) *Let m be the number of outcomes favorable to the occurrence of an event A. Let n be the number of exhaustive outcomes. The probability of event A occurring, denoted by P(A) and defined by,*

$P(A) = \frac{m}{n} = \frac{\text{Number of outcomes favorable to A}}{\text{Number of exhaustive outcomes}}$

1.3 Axioms of probability

If S is the sample space and E is any event in a random experiment,

(1)$P(S) = 1$

(2)$0 \leq P(E) \leq 1$

(3) For two events E_1 and E_2 with $E_1 \cap E_1 = \phi$

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$

Definition 9 (Probability of compliment event) *If E is a event and P(E) is the probability of a event E, then the probability of compliment of E (say E′) is denoted by P(E′) and defined by* $P(E') = 1 - P(E)$.

Example 1 *Three unbiased coins are tossed. Find the probability of getting (i) exactly two heads, (ii) at least one tail, (iii) at most two heads (iv) a head on the second coin, and (v) exactly two heads in succession.*

Solution:

The sample space S is given as following when three coins are tossed

S={HHH, HTH, THH, HHT, TTT, THT, TTH, HTT }

$n(S) = 8$

(i) Let A be the event getting exactly two heads.

A={HTH, THH, HHT }

$n(A) = 3$

$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$

(ii) Let B be the event getting at least one tail.

B={HTH, THH, HHT, TTT, THT, TTH, HTT }

$n(B) = 7$

$P(B) = \frac{n(B)}{n(S)} = \frac{7}{8}$

(iii) Let C be the event getting at most two heads.

C={HTH, THH, HHT, TTT, THT, TTH, HTT }

$n(C) = 7$

$P(C) = \frac{n(C)}{n(S)} = \frac{7}{8}$

(iv) Let D be the event getting a head on the second coin.

D={HHH, THH, HHT, THT }

$n(D) = 4$

$P(D) = \frac{n(D)}{n(S)} = \frac{4}{8} = \frac{1}{2}$

(v) Let E be the event getting two heads in succession.

D={HHH, THH, HHT }

$n(E) = 3$

$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$

Example 2 *What is the chance that a leap year selected at random will contain 53 Sundays?*

Solution:

In a leap year (which consists of 366 days) there are 52 complete weeks and 2 days over. The following are the possible combinations for these two over days:

(i) Sunday and Monday, (ii) Monday and Tuesday, (iii) Tuesday and Wednesday, (iv) Wednesday and Thursday, (v) Thursday and Friday, (vi) Friday and Saturday and (vii) Saturday and Sunday.

Required probability $= \frac{2}{7}$

Example 3 *A fair dice is thrown. Find the probability of getting (i) an even number, (ii) a perfect square, (iii) an integer greater than or equal to 3.*

Solution:

The sample space S is given as following when a dice is thrown

S={1, 2, 3, 4, 5, 6 }

$n(S) = 6$

(i) Let A be the event getting an even number.

A={2, 4, 6}

$n(A) = 3$

$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6}$

(ii) Let B be the event getting a perfect square.

B={1, 4}

$n(B) = 2$

$P(B) = \frac{n(B)}{n(S)} = \frac{2}{6}$

(iii) Let C be the event getting an integer greater than or equal to 3.

C={3, 4, 5, 6}

$n(C) = 4$

$P(C) = \frac{n(C)}{n(S)} = \frac{4}{6} = \frac{2}{3}$

Example 4 *A card is drawn from a pack of 52 cards(well-shuffled). Find the probability of (i) getting a king card, (ii) getting a face card, (iii) getting a red card (iv) getting a card between 2 and 7, both inclusive, and (v) getting a card between 2 and 8, both exclusive.*

Solution:

Total number of cards is 52, one card out of 52 cards can be drawn i ways.

$n(S) =^{52} C_1 = 52$ (i) Let A be the event of getting a king kard. There are 4 king cards and one of them can be drawn in ${}^4C_1$ ways.

$n(A) =^4 C_1 = 4$

$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$

(ii) Let B be the event of getting a face card. There are 12 face cards and one of them can be drawn in ${}^{12}C_1$ ways.

$n(B) =^{12} C_1 = 12$

$P(B) = \frac{n(B)}{n(S)} = \frac{12}{52} = \frac{3}{13}$

(iii) Let C be the event of getting a red card. There are 26 red cards and one of them can be drawn in ${}^{26}C_1$ ways.

$n(C) =^2 6C_1 = 26$

$P(C) = \frac{n(C)}{n(S)} = \frac{26}{52} = \frac{1}{2}$

(iv) Let D be the event of getting a card between 2 and 7(both inclusive). There are $6 \times 4 = 24$ such cards and one of them can be drawn in ${}^{24}C_1$ ways.

$n(D) =^{24} C_1 = 24$

$P(D) = \frac{n(D)}{n(S)} = \frac{24}{52} = \frac{6}{13}$

(v) Let E be the event of getting a card between 2 and 8(both exclusive). There are $5 \times 4 = 20$ such cards and one of them can be drawn in ${}^{20}C_1$ ways.

$n(E) =^2 0C_1 = 20$

$P(E) = \frac{n(E)}{n(S)} = \frac{20}{52} = \frac{5}{13}$

Example 5 *A bag contains 2 black ball, 3 red, and 5 blue balls. Three balls are drawn at random. Find the probability that tha e three balls drawn(i) are blue, (ii) consist of 2 blue and 1 red ball, and (iii) consist of exactly one black ball.*

Solution:

Total number of balls are 10, out of them three balls can be drawn in ${}^{10}C_3$ ways.

$n(S) = {}^{10}C_3 = 120$ (i) Let A be the event of three balls are blue. 3 blue balls out of 5 blue balls can be drawn in ${}^5C_3$ ways.

$n(A) = {}^5C_3 = 10$

$P(A) = \frac{n(A)}{n(S)} = \frac{10}{120} = \frac{1}{12}$

(ii) Let B be the event that the three balls drawn consist of 2 blue and 1 red ball. 2 blue balls out of 5 balls can be drawn in ${}^5C_2$ ways. 1 red ball out of 3 red balls can be drawn in ${}^3C_1$ ways. $n(B) = {}^5C_2 \times {}^3C_1 = 30$

$P(B) = \frac{n(B)}{n(S)} = \frac{30}{120} = \frac{1}{4}$

(iii) Let C be the event that the three balls drawn consist one black ball(remaining 2 can be drawn from 3 red and 5 blue balls). 1 black ball out of 2 balls can be drawn in ${}^2C_1$ ways. $n(C) = {}^2C_1 \times {}^8C_2 = 56$

$P(C) = \frac{n(C)}{n(S)} = \frac{56}{120} = \frac{7}{15}$

Theorem 1 *For any two events A and B in a sample space S,*
$P(\bar{A} \cap B) = P(B) - P(A \cap B)$

Theorem 2 *The probability that at least one events A and B will occur is given by,*
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Theorem 3 *The probability that at least one events A, B and C will occur is given by,*
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

Example 6 *Two cards are drawn from a pack of cards. find the probability that they will be both red or both pictures.*

Solution:

Let A and B be the events that both cards drawn are red and pictures respectively.

$$P(A) = \frac{^{26}C_2}{^{52}C_2} = \frac{325}{1326}$$
$$P(B) = \frac{^{12}C_2}{^{52}C_2} = \frac{66}{1326}$$
$$P(A \cap B) = \frac{^{6}C_2}{^{52}C_2} = \frac{15}{1326}$$

Probability that both cards drawn are red or pictures

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{325}{1326} + \frac{66}{1326} - \frac{15}{1326} \\ &= \frac{188}{663} \end{aligned}$$

Example 7 *A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.*

Solution:

Let A, B and C be the events that the card drawn is a king, a heart and a red card respectively.

$$P(A) = \frac{^{4}C_1}{^{52}C_1} = \frac{4}{52}$$
$$P(B) = \frac{^{13}C_1}{^{52}C_1} = \frac{13}{52}$$
$$P(C) = \frac{^{26}C_1}{^{52}C_1} = \frac{26}{52}$$
$$P(A \cap B) = \frac{^{1}C_1}{^{52}C_1} = \frac{1}{52}$$
$$P(B \cap C) = \frac{^{13}C_1}{^{52}C_1} = \frac{13}{52}$$
$$P(A \cap C) = \frac{^{2}C_1}{^{52}C_1} = \frac{2}{52}$$
$$P(A \cap B \cap C) = \frac{^{1}C_1}{^{52}C_1} = \frac{1}{52}$$

Probability that both cards drawn are red or pictures

$$\begin{aligned}P(A\cup B\cup C) =& P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)\\ =& \frac{4}{52}+\frac{13}{52}+\frac{26}{52}-\frac{1}{52}-\frac{13}{52}-\frac{2}{52}+\frac{1}{52}\\ =& \frac{7}{3}\end{aligned}$$

1.4 Conditional probability

If A and B are two events in a sample space S, the probability of their simultaneous occurrence,

$P(A\cap B) = P(A)P(B/A)$ or $P(A\cap B) = P(B)P(A/B)$

where $P(A/B)$ is the conditional probability of B given that A has already occurred. $P(B/A)$ is the conditional probability of A given that B has already occurred

Theorem 4 *If A and B are two independent events, the probability of their simultaneous occurrence is given by*

$P(A\cap B) = P(A)P(B)$

Theorem 5 *If A and B are independent events then $\bar{A}$ and $\bar{B}$ are also independent events.*

Example 8 *Find the probability of drawing a queen and a king froma pack of cards in two consecutive draws, the cards not being replaced.*

Solution:

Let A be the event that the card drawn is a queen.

$P(A) = \frac{^4C_1}{^{52}C_1} = \frac{4}{52} = \frac{1}{13}$

Let B be the event that the cards drawn are a king in the second draw given that the first card drawn is a queen.

$P(B/A) = \frac{^4C_1}{^{51}C_1} = \frac{4}{51}$

Probability that the cards drawn are a queen and a king

$$\begin{aligned}P(A\cap B) =& P(A)P(B/A)\\ =& \frac{4}{52}\times\frac{4}{51}\\ =& \frac{4}{663}\end{aligned}$$

1.5 Baye's theorem

Let $A_1, A_2, \ldots, A_n$ be n mutually exclusive and exhaustive events with $P(A_i) \neq 0$, $i = 1, 2, \ldots, n$ in a sample space S. let B be an event that can occur in combination with any one of the events $A_1, A_2, \ldots, A_n$ with $P(B) \neq 0$. The probability of the event A_i when the event B has actually occurred is given by

$$P(A_i/B) = \frac{P(A_i)P(B/A_i)}{\sum\limits_{i=1}^{n} P(A_i)P(B/A_i)}$$

Example 9 *A company has two plants to manufacture hydraulic machines. Plant I manufactures 70% of the hydraulic machines, and Plant II manufactures 30%. At Plant I, 80% of the hydraulic machines are rated standard quality. At Plant II, 90% of the hydraulic machines are rated standard quality. A machine picked up at random and is found to b of standard quality. What is the probability that it has come from Plant I?*

Solution:

Let E_1 and E_2 be the hydraulic machines are manufactured in Plant I and Plant II respectively. Let B the event that the machine picked up is found to be of standard quality.

$P(E_1) = \frac{70}{100} = 0.7$

$P(E_2) = \frac{30}{100} = 0.3$

Probability that the machine is of standard quality given that it is manufactured in Plant I

$P(B/E_1) = \frac{80}{100} = 0.8$

Probability that the machine is of standard quality given that it is manufactured in Plant II

$P(B/E_1) = \frac{90}{100} = 0.9$

Probability that the a machine is manufactured in Plant I given it is of standard quality

$$\begin{aligned} P(E_1/B) =& \frac{P(E_1)P(B/E_1)}{P(E_1)P(B/E_1) + P(E_2)P(B/E_2)} \\ =& \frac{0.7 \times 0.8}{0.7 \times 0.8 + 0.3 \times 0.9} \\ =& 0.6747 \end{aligned}$$

Example 10 *The contents of urns I, II and III are as follows:*
1 white, 2 black and 3 red balls, 2 white, 1 black and 1 red balls, and 4 white, 5 black and 3 red balls. One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urns I, II or III ?

Solution:

LetE_1, E_2 and E_3 denote the events that the urn I, II and III is chosen, respectively, and let A be the event that the two balls taken from the selected urn are white and red. Then
$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$
$P(A/E_1) = \frac{1\times3}{^6C_2} = \frac{1}{5}$, $P(A/E_2) = \frac{2\times1}{^4C_2} = \frac{1}{3}$ and $P(A/E_3) = \frac{4\times3}{^{12}C_2} = \frac{2}{11}$
probability that they come from urns I, II or III

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$
$$= \frac{\frac{1}{3}\times\frac{1}{3}}{\frac{1}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{2}{11}} = \frac{55}{103}$$

Similarly,

$$P(E_3/A) = \frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$
$$= \frac{\frac{1}{3}\times\frac{2}{11}}{\frac{1}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{2}{11}} = \frac{30}{103}$$

Hence,

$$P(E_1/A) = 1 - \frac{55}{103} - \frac{30}{103} = \frac{18}{103}$$

Example 11 *In a bolt factory machines A,B and C manufacture respectively 25%.35% and 40% of the total. Of their output 5, 4, and 2 percent are defective bolts. A bolts drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?*

Solution:

Let E_1, E_2 and E_3, denote the events that a bolt-selected at random is manufactured by the machines A, B and C respectively, and let E denote the event of its being defective. Then we have

$P(E_1) = 0.25,\ P(E_2) = 0.35,\ P(E_3) = 0.40$

The probability of drawing a defective bolt manufactured by machine A is

$P(E/E_1) = 0.05$

similarly, we have

$P(E/E_2) = 0.04$ and $P(E/E_3) = 0.02$

Hence the probability t at a defective bolt selected-at random is manufactured by machine A is given by

$$\begin{aligned}P(E_1/E) =& \frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2) + P(E_3)P(E/E_3)}\\ =& \frac{0.25 \times 0.05}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = \frac{125}{345} = \frac{25}{69} = 0.3623\end{aligned}$$

Similarly

$$\begin{aligned}P(E_2/E) =& \frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2) + P(E_3)P(E/E_3)}\\ =& \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = \frac{140}{345} = \frac{28}{69} = 0.4058\end{aligned}$$

and

$$P(E_3/E) = 1 - P(E_1/E) - P(E_2/E) = 1 - \frac{25}{69} - \frac{28}{69} = \frac{16}{69} = 0.2319$$

2 Random Variables

Definition 10 (Random variables) *Intuitively by a random variable (r.v.) we mean a real number X connected with the outcome of a random experiment E.*

For example, if E consists of two tosses of a coin, we may consider the random variable which is the number of heads (0, 1 or 2).

Outcome	HH	HT	TH	TT
Value of X	2	1	1	0

2.1 Discrete Random Variables

If a random,variable takes at most a countable number of values, it is called a discrete random variable. In other words, a real valued function defined on a discrete sample space is called a discrete random variable.

2.2 Probability Mass Function

Suppose X is a discrete random variable takes the values $x_1, x_2, \ldots, x_n$ With each possible outcome x_i. The probability of x_i is $p_i = P(X = x_i) = p(x_i)$. The numbers $p(x_i)i, 1, 2, \ldots$ must satisfy the following conditions:

(i) $p(x_i) \geq 0 \ \forall i$

(ii) $\sum\limits_{i=1}^{n} p(x_i) = 1$

This function $p(x_i)$ is called the probability mass function of the random variable X and the set $(x_i, p(x_i))$ is called the probability distribution (p.d.) of the r.v. X.

2.3 Discrete Distribution Function

Let X be a discrete random variable from range of the values $x_1, x_2, \ldots$ with $x_1 < x_2 < \ldots$ with probabilities $p(x_1), p(x_1) \ldots$ $(p(x_i) \geq 0, \ \forall i)$ and $\sum\limits_{i=1}^{x} p(x_i) = 1$.

The function F(x) of the disrete random variable X is defined on a integer x as,

$F(x) = P(X \leq x) = \sum\limits_{i=1}^{x} p(x_i)$ where the function F(x) is called the cumulative distribution function. The pairs $\{x_i, F(x)\}$, i=1,2,..is called the cumulative probability function.

Example 12 *Find the probability distribution of the number of heads when three coins are tossed.*

Solution:

When three coins are tossed,

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Let X be the random variable of getting heads in tossing of three coins and it takes the values 0, 1, 2, 3.

P(X=0)=P(no head)=$\frac{1}{8}$

P(X=1)=P(one head)=$\frac{3}{8}$

P(X=2)=P(two heads)=$\frac{3}{8}$

P(X=3)=P(three heads)=$\frac{1}{8}$

we have, $\sum P(X=x)=\frac{1}{8}+\frac{3}{8}+\frac{3}{8}+\frac{1}{8}=1$

The probability distribution of X is

$X=x$	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Example 13 *A fair die is tossed once. If the random variable is getting an even number, find the probability distribution of X.*

Solution:

When a fair die is tossed,

$S=\{1,2,3,4,5,6\}$

Let X be the random variable of getting getting an even number, hence it takes the values 0 and 1

P(X=0)=P(1, 3, 5)=$\frac{3}{6}=\frac{1}{2}$

P(X=1)=P(2 ,4, 6)=$\frac{3}{6}=\frac{1}{2}$

we have, $\sum P(X=x)=\frac{1}{2}+\frac{1}{2}=1$

The probability distribution of X is

$X=x$	0	1
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}$

Example 14 *State with reasons whether the following represent the probability mass function of a random variable.*

(i)

$X=x$	0	1	2	3
$P(X=x)$	0.4	0.3	0.2	0.1

(ii)

$X=x$	0	1	2	3
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$

(iii)

$X=x$	0	1	2	3
$P(X=x)$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$

Solution:

(i) Here, $0 \leq P(X=x) \leq 1$ is satisfied for all values of X.

$$\begin{aligned}\sum P(X=x) =& P(X=0)+P(X=1)+P(X=2)+P(X=3)\\ =& 0.4+0.3+0.2+0.1\\ =& 1\end{aligned}$$

Here we get $P(X=x)=1$. Hence it is represent probability mass function.

(ii) Here, $0 \leq P(X=x) \leq 1$ is satisfied for all values of X.

$$\begin{aligned}\sum P(X=x) =& P(X=0)+P(X=1)+P(X=2)+P(X=3)\\ =& \frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\frac{1}{4}\\ =& \frac{5}{4} > 1\end{aligned}$$

Here we get $P(X=x) > 1$. Hence it does not represent a probability mass function.

(iii) Here, $0 \leq P(X=x) \leq 1$ is not satisfied for all values of X as $P(X=0) = -\frac{1}{2}$.

Hence $P(X=x)$ does not represent a probability mass function.

Example 15 *A random variable X has the following probability distribution,*

X	0	1	2	3	4	5	6	7
$P(X=x)$	a	4a	3a	7a	8a	10a	6a	9a

(i) Find the value of a.

(ii) Find $P(X<3)$.

(iii) Find the smallest value of m for which $P(X \leq m) \geq 0.6$.

Solution:

(i) Since, $P(X = x)$ is a probability distribution function, $\sum P(X = x) = 1$

$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) = 1$

$a + 4a + 3a + 7a + 8a + 10a + 6a + 9a = 1$

$a = \frac{1}{48}$

(ii)

$$\begin{aligned} P(X < 3) =& P(X = 0) + P(X = 1) + P(X = 2) \\ =& a + 4a + 3a \\ =& 8a \\ =& 8(\frac{1}{48}) \\ =& \frac{1}{6} \end{aligned}$$

$$= 0.16$$

(iii)

$$\begin{aligned} P(X \leq 4) =& P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) \\ =& a + 4a + 3a + 7a + 8a \\ =& 23a \\ =& 23(\frac{1}{48}) \\ =& \frac{23}{48} \\ =& 0.575 \\ P(X \leq 5) =& P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\ =& a + 4a + 3a + 7a + 8a + 10a \\ =& 33a \\ =& 33(\frac{1}{48}) \\ =& \frac{33}{48} \\ =& 0.69 \end{aligned}$$

Hence, the smallest value of m for which $P(X \leq m) \geq 0.6$ is 5.

Example 16 *Construct the distribution functon of the discrete random variable X whose probability distribution is as given below:*

X	1	2	3	4	5	6	7
$P(X=x)$	0.1	0.15	0.25	0.2	0.15	0.1	0.05

Solution:

Distribution function of X

X	$P(X=x)$	$F(x)$
1	0.1	0.1
2	0.15	0.25
3	0.25	0.5
4	0.2	0.7
5	0.15	0.85
6	0.1	0.95
7	0.05	1

2.4 Continuous Random variables

If a random,variable takes any values in a given interval, it is called a continuous random variable. In other words, a real valued function defined on a continuous sample space is called a continuous random variable.

2.5 Probability Density Function

Suppose X is a continuous random variable such that the probability of the variable X falling in the small interval $x - \frac{dx}{2}$ to $x + \frac{dx}{2}$ is $f(x)$, i.e.,

$p(x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}) = f(x)dx$

The function $f(x)$ is called probability density function of the random variable X

Properties of probability density function

(1) $f(x) \geq 0, -\infty < x < \infty$

(2) $\int_{-\infty}^{\infty} f(x)dx = 1$

(3) $P(a < x < b) = \int_{a}^{b} f(x)dx$

2.6 Continuous Distribution Function

The probability density function $f(x)$ defined on a continuous random variable then the cumulative distribution function or distribution function is defined as follow,

$F(X) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx, \ -\infty < x < \infty$

The function $F(x)$ satisfied the following properties,

(1) $F(-\infty) = 0$

(2) $F(\infty) = 1$

(3) $0 \leq F(x) \leq 1, \ -\infty < x < \infty$

(4) $P(a < X < b) = F(b) - F(a)$

Example 17 *Is the function $f(x)$ defined as*

$$f(x) = \frac{1}{7} \quad 1 < x < 8$$
$$= 0 \quad otherwise$$

is a probability density function for a r.v. If yes then find $P(3 < X < 10)$

Solution:

Value of $f(x)$ is non-zero in $1 < x < 8$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{1} f(x)dx + \int_{1}^{8} f(x)dx + \int_{8}^{\infty} f(x)dx$$
$$= 0 + \int_{1}^{8} \frac{1}{7} dx + 0$$
$$= \frac{1}{7}[x]_1^8 = \frac{1}{7}(8 - 1)$$
$$= 1$$

Therefore, $f(x)$ is a probability density function.

$$
\begin{aligned}
P(3 < X < 10) &= \int_3^{10} f(x)dx \\
&= \int_3^{8} f(x)dx + \int_8^{10} f(x)dx \\
&= \int_3^{8} \frac{1}{7}dx + 0 \\
&= \frac{1}{7}[x]_3^8 = \frac{1}{7}(8-3) \\
&= \frac{5}{7}
\end{aligned}
$$

Example 18 *If the probability density function $f(x)$ defined as,*

$$
\begin{aligned}
f(x) =& kx^2 \quad 0 < x < 3 \\
=& 0 \quad otherwise
\end{aligned}
$$

then find the value of k and compute (i) $P(1 < X < 2)$ (ii) $P(X < 2)$ and (iii)$P(X \geq 2)$.

Solution:

Given that $f(x)$ is a probability density function,

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx =& 1 \\
\int_{-\infty}^{0} f(x)dx + \int_0^{3} f(x)dx + \int_3^{\infty} f(x)dx =& 1 \\
0 + \int_0^{3} kx^2 dx + 0 =& 1 \\
k[\frac{x^3}{3}]_0^3 =& 1 \\
\frac{k}{3}(27-0) =& 1 \\
9k =& 1 \\
k =& \frac{1}{9}
\end{aligned}
$$

Therefore,

$$f(x) = \frac{1}{9}x^2, \quad 0 < x < 3$$
$$= 0 \quad \text{otherwise}$$

(i)

$$\begin{aligned}
P(1 < X < 2) &= \int_1^2 f(x)dx \\
&= \int_1^2 \frac{1}{9}x^2 dx \\
&= \frac{1}{9}\left[\frac{x^3}{3}\right]_1^2 \\
&= \frac{1}{27}(8-1) \\
&= \frac{7}{27}
\end{aligned}$$

(ii)

$$\begin{aligned}
P(X < 2) &= \int_{-\infty}^2 f(x)dx \\
&= \int_{-\infty}^0 f(x)dx + \int_0^2 \frac{1}{9}x^2 dx \\
&= 0 + \frac{1}{9}\left[\frac{x^3}{3}\right]_0^2 \\
&= \frac{1}{27}(8-0) \\
&= \frac{8}{27}
\end{aligned}$$

(iii)

$$\begin{aligned}
P(X \geq 2) &= 1 - P(X < 2) \\
&= 1 - \frac{8}{27} \\
&= \frac{19}{27}
\end{aligned}$$

2.7 Two Dimensional Discrete Random Variable

Let X and Y be two random variables defined on the same sample space S, then the function (X,Y) that assigns a point in R^2 is called a two-dimensional random variable. A two dimensional random variable is said to be discrete if it takes at most a countable number of points in R^2. The possible values of a two dimensional discrete random variable (X,Y) may be represented as (x_i, x_j), $i = 1, 2, 3, ..., m, ...$ and $j = 1, 2, 3, ..., n, ...$

2.7.1 Joint probability mass function

If (X,Y) is a two dimensional discrete random variable, then the joint discrete function or joint probability mass function of X, Y, denoted by p_{XY} and defined by

$p_{XY}(x_i, x_j) = P(X = x_i, Y = x_j)$ for a value of (x_i, x_j) of (X, Y).

$p_{XY}(x_i, x_j) = 0$, otherwise.

The probability mass function p_{XY} satisfies the following conditions,
(i) $p_{XY}(x_i, x_j) \geq 0$, for all i and j.
(ii) $\sum_{j=1}^{n} \sum_{i=1}^{m} p_{XY}(x_i, x_j) = 1$.

2.7.2 Cumulative Distribution Function

If (X,Y) is a two dimensional discrete random variable, then $F_{XY}(x, y) = P(X \leq x, Y \leq y)$ is called the cumulative distribution function (cdf) of (X,Y) and defined by

$$F_{XY}(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{m} p_{XY}(x_i, x_j) = \sum \sum p_{ij}$$

2.7.3 Marginal Probability Function

If (X,Y) is a two dimensional discrete random variable which takes up countable number of values (x_i, x_j). Then the probability distribution of X is given by

$$\begin{aligned} p_X(x_i) =& P(X = x_i) \\ =& (X = x_i, Y = y_1) + (X = x_i, Y = y_2) + \ldots + (X = x_i, Y = y_m) \\ =& p_{i1} + p_{i2} + \ldots + p_{im} \\ =& \sum_{j=1}^{m} p_{ij} \\ =& \sum_{j=1}^{m} p(x_i, x_j) = p_i^* \end{aligned}$$

and it is known as marginal probability mass or discrete marginal density function of X. Similarly,

$$p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = \sum_{i=1}^{n} p(x_i, x_j) = p_j^*$$

is the marginal probability mass function of Y.

2.7.4 Conditional Probability Function

If (X,Y) is a two dimensional discrete random variable. Then the conditional discrete density function or conditional probability mass function of X, given $Y = y$, denoted by $p_{X/Y}(x/y)$ is defined by

$$p_{X/Y} = P(X = x/Y = y) = \frac{P(X = x/Y = y)}{P(Y = y)}$$

, provided $P(Y = y) \neq 0$.

The conditional discrete density function or conditional probability mass function of Y, given $X = x$, denoted by $p_{Y/X}(y/x)$ is defined by

$$p_{Y/X} = P(Y = y/X = x) = \frac{P(X = x/Y = y)}{P(X = x)}$$

, provided $P(X = x) \neq 0$.

Example 19 *From the following table for bivariate distribution of (X,Y), find (i) $P(X \leq 1)$ (ii) $P(Y \leq 3)$ (iii) $P(X \leq 1, Y \leq 3)$ (iv) $P(X \leq 1/Y \leq 3)$ (v) $P(Y \leq 3/X \leq 1)$ (vi) $P(X + Y \leq 4)$*

$X \downarrow$ $\quad Y \rightarrow$	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution:

Marginal distribution

$X \downarrow$ $\quad Y \rightarrow$	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
							$\sum p(x) = 1$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\sum p(y) = 1$

(i)

$$
\begin{aligned}
P(X \leq 1) =& P(X = 0) + P(X = 1) \\
=& \frac{8}{32} + \frac{10}{16} \\
=& \frac{7}{8}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
P(Y \leq 3) =& + P(Y = 1) + P(Y = 2) + P(Y = 3) \\
=& \frac{3}{32} + \frac{3}{32} + \frac{11}{64} \\
=& \frac{23}{64}
\end{aligned}
$$

(iii)

$$\begin{aligned}P(X \leq 1, Y \leq 3) =& P(X=0,Y=1)+P(X=0,Y=2)+P(X=0,Y=3)+P(X=1,Y=1)\\ &+P(X=1,Y=2)+P(X=1,Y=3)\\ =&0+0+\frac{1}{32}+\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\\ =&\frac{9}{32}\end{aligned}$$

(iv)

$$\begin{aligned}P(X \leq 1/Y \leq 3) =& \frac{P(X \leq 1, Y \leq 3)}{P(Y \leq 3)}\\ =&\frac{\frac{9}{32}}{\frac{23}{64}}\\ =&\frac{18}{23}\end{aligned}$$

(v)

$$\begin{aligned}P(Y \leq 3/X \leq 1) =& \frac{P(X \leq 1, Y \leq 3)}{P(X \leq 1)}\\ =&\frac{\frac{9}{32}}{\frac{7}{8}}\\ =&\frac{9}{28}\end{aligned}$$

(vi)

$$\begin{aligned}P(X+Y \leq 4) =& P(X=0,Y=1)+P(X=0,Y=2)+P(X=0,Y=3)+P(X=0,Y=4)\\ &+P(X=1,Y=1)+P(X=1,Y=2)+P(X=1,Y=3)+P(X=2,Y=1)\\ &+P(X=2,Y=2)\\ =&0+0+\frac{1}{32}+\frac{2}{32}+\frac{1}{16}+\frac{1}{16}+\frac{1}{8}+\frac{1}{32}+\frac{1}{32}\\ =&\frac{13}{32}\end{aligned}$$

2.8 Two Dimensional Continuous Random Variable

2.8.1 Joint probability density function

If (X,Y) is a two dimensional continuous random variable, such that $P\{x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2}$ and $y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2}\} = f(x,y)dxdy$, then $f(x,y)$ is called the joint probability density function of (x,y).

The probability density function $f(x, y)$ satisfies the following conditions,

(i) $f(x, y) \geq 0$, for all x and y.

(ii)$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$

(iii)$P(a \leq X \leq b, c \leq Y \leq d) = \int_{a}^{b} \int_{c}^{d} f(x, y)dxdy$

2.8.2 Cumulative distribution function

If (X, Y) is a two dimensional continuous random variable, then $F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y)dxdy$ is called the cumulative distribution function(CDF) of (x, y).

2.8.3 Marginal probability function

If (X, Y) is a two dimensional continuous random variable which assumes all the values in a special region R in the xy-plane. Then the probability distribution of X is given by $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$ and is called as the marginal probability density function of X. Similarly, $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$ and is called as the marginal probability density function of Y.

2.8.4 Conditional probability function

If (X, Y) is a two dimensional continuous random variable. Then the conditional continuous density function or conditional probability density function of X, given $Y = y$, denoted by $f(x/y)$ is defined as

$$f(x/y) = \frac{f(x, y)}{f_Y(y)}$$

The conditional probability density function of Y, given $X = x$, denoted by $f(y/x)$ is defined as

$$f(y/x) = \frac{f(x, y)}{f_X(x)}$$

A necessary and sufficient condition for the continuous random variable X and Y to be independent is

$$f(x,y) = f_X(x) f_Y(y)$$

.

Example 20 *Two random variables X and Y have the joint probability density function*

$$f(x,y) = Ae^{-(2x+y)}, \quad x,y \geq 0$$
$$= 0, \quad otherwise$$

Find (i) A (ii) marginal pdf of X and Y (iii) f(y/x)

Solution:

(i) Since $f(x,y)$ is a pdf,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$
$$\int_{0}^{\infty}\int_{0}^{\infty} Ae^{-(2x+y)}dxdy = 1$$
$$A\int_{0}^{\infty}\left[\int_{0}^{\infty} e^{-2x}dx\right] e^{-y}dy = 1$$
$$A\int_{0}^{\infty}\left|\frac{e^{-2x}}{-2}\right|_{0}^{\infty} e^{-y}dy = 1$$
$$\frac{A}{-2}\int_{0}^{\infty}\left(e^{-\infty} - e^{0}\right) e^{-y}dy = 1$$
$$\frac{A}{-2}\int_{0}^{\infty}(-1)\, e^{-y}dy = 1$$
$$\frac{A}{2}\int_{0}^{\infty} e^{-y}dy = 1$$
$$\frac{A}{2}\left|-e^{-y}\right|_{0}^{\infty} = 1$$
$$\frac{-A}{2}\left(e^{-\infty} - e^{0}\right) = 1$$
$$\frac{A}{2} = 1$$
$$A = 2$$

(ii)

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_0^{\infty} Ae^{-(2x+y)}dy \\ &= -2\left|e^{-(2x+y)}\right|_0^{\infty} \\ &= -2\left(e^{-\infty} - e^{-2x}\right) \\ &= 2e^{-2x}, \quad x \geq 0 \end{aligned}$$

Therefore,

$$\begin{aligned} f_X(x) &= 2e^{-2x}, \quad x \geq 0 \\ &= 0, \quad x < 0. \end{aligned}$$

Now,

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \int_0^{\infty} Ae^{-(2x+y)}dx \\ &= \left|e^{-(2x+y)}\right|_0^{\infty} \\ &= \left(-e^{-\infty} + e^{-y}\right) \\ &= e^{-y}, \quad y \geq 0 \end{aligned}$$

Therefore,

$$\begin{aligned} f_Y(y) &= 2e^{-y}, \quad y \geq 0 \\ &= 0, \quad y < 0. \end{aligned}$$

(iii)

$$\begin{aligned} f(y/x) &= \frac{f(x,y)}{f_X(x)} \\ &= \frac{2e^{-(2x+y)}}{2e^{-2x}} \\ &= e^{-y}, \quad y \geq 0 \end{aligned}$$

Example 21 *If the joint distribution function of X and Y is given by*

$$F(x,y) = (1-e^{-x})(1-e^{-y}), \quad x>0, y>0$$
$$= 0, \quad otherwise.$$

(i) Find $f_X(x), f_Y(y)$. (ii) Are X and Y are independent. (iii) Find $P(1<X<3, 1<Y<2)$.

Solution:

Here,

$$F(x,y) = (1-e^{-x})(1-e^{-y})$$

Therefore, the joint pdf is given by

$$\begin{aligned} f(x,y) &= \frac{\partial^2 F}{\partial x \partial y} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(1-e^{-x})(1-e^{-y})\right) \\ &= e^{-y}\frac{\partial}{\partial x}(1-e^{-x}) \\ &= e^{-(x+y)}, \quad x>0, y>0 \end{aligned}$$

Therefore,

$$f(x,y) = e^{-(x+y)}, \quad x>0, y>0,$$
$$= 0, \quad otherwise.$$

(i)

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_0^{\infty} e^{-(x+y)}dy \\ &= \left|-e^{-(x+y)}\right|_0^{\infty} \\ &= (-e^{-\infty} + e^{-x}) \\ &= e^{-x}, \quad x>0. \end{aligned}$$

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\
&= \int_{0}^{\infty} e^{-(x+y)}dx \\
&= \left|-e^{-(x+y)}\right|_0^{\infty} \\
&= (-e^{-\infty} + e^{-y}) \\
&= e^{-y}, \quad y > 0.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
f_X(x)f_Y(y) &= e^{-x}e^{-y} \\
&= e^{-(x+y)} \quad x > 0, y > 0 \\
&= f(x,y)
\end{aligned}
$$

Hence, X and Y are independent. (iii) Since, X and Y are independent,

$$
\begin{aligned}
P(1 < X < 3, 1 < Y < 2) &= P(1 < X < 3)P(1 < Y < 2) \\
&= \int_1^3 f_X(x)dx \int_1^2 f_Y(y)dy \\
&= \int_1^3 e^{-x}dx \int_1^2 e^{-y}dy \\
&= \left|-e^{-x}\right|_1^3 \left|-e^{-y}\right|_1^2 \\
&= (-e^{-3} + e^{-1})(-e^{-2} + e^{-1}) \\
&= e^{-5} - e^{-4} - e^{-3} - e^{-2}.
\end{aligned}
$$

Example 22 *The joint pdf of (X, Y) is given by*

$$
f(x,y) = \frac{1}{4}e^{-|x|-|y|}, \quad -\infty < x < \infty, -\infty < y < \infty.
$$

(i) Are X and Y independent? (ii) Find the probability that $X \leq 1$ and $Y < 0$.

Solution:

We know that,

$$
\begin{aligned}
|x| &= -x, \quad -\infty < x \leq 0 \\
&= x, \quad 0 \leq x < \infty
\end{aligned}
$$

$$|y| = -y, \quad -\infty < y \leq 0$$
$$= y, \quad 0 \leq y < \infty$$

Therefore,

$$\begin{aligned} f(x,y) &= \frac{1}{4}e^{-|x|-|y|}, \quad -\infty < x < \infty, -\infty < y < \infty \\ &= \frac{1}{4}e^{x+y}, \quad -\infty < x \leq 0, -\infty < y \leq 0, \\ &= \frac{1}{4}e^{-x-y}, \quad 0 \leq x < \infty, 0 \leq y < \infty. \end{aligned}$$

(i)

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{-\infty}^{\infty} \frac{1}{4}e^{-|x|-|y|}dy \\ &= e^{-|x|}\int_{-\infty}^{\infty} \frac{1}{4}e^{-|y|}dy \\ &= \frac{e^{-|x|}}{4}\left(\int_{-\infty}^{0} e^{y}dy + \int_{0}^{\infty} e^{-y}dy\right) \\ &= \frac{e^{-|x|}}{4}\left(|e^{y}|_{-\infty}^{0} - |e^{-y}|_{0}^{\infty}\right) \\ &= \frac{e^{-|x|}}{4}\left((e^{0} - e^{-\infty}) - (e^{-\infty} - e^{0})\right) \\ &= \frac{e^{-|x|}}{4}(1+1) \\ &= \frac{e^{-|x|}}{2} \quad -\infty < x < \infty \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \int_{-\infty}^{\infty} \frac{1}{4}e^{-|x|-|y|}dx \\ &= e^{-|y|}\int_{-\infty}^{\infty} \frac{1}{4}e^{-|x|}dx \\ &= \frac{e^{-|y|}}{4}\left(\int_{-\infty}^{0} e^{x}dx + \int_{0}^{\infty} e^{-x}dx\right) \\ &= \frac{e^{-|y|}}{4}\left(|e^{x}|_{-\infty}^{0} - |e^{-x}|_{0}^{\infty}\right) \\ &= \frac{e^{-|y|}}{4}\left((e^{0} - e^{-\infty}) - (e^{-\infty} - e^{0})\right) \\ &= \frac{e^{-|y|}}{4}(1+1) \\ &= \frac{e^{-|y|}}{2} \quad -\infty < y < \infty \end{aligned}$$

Now,

$$
\begin{aligned}
f_X(x)f_Y(y) &= \frac{e^{-|x|}}{2}\frac{e^{-|y|}}{2} \\
&= \frac{e^{-(|x|+|y|)}}{4} \qquad -\infty < x < \infty, -\infty < y < \infty \\
&= f(x,y)
\end{aligned}
$$

Hence, X and Y are independent.

(ii)

$$
\begin{aligned}
P(X \leq 1, Y < 0) &= \int_{-\infty}^{0}\int_{-\infty}^{1} f(x,y)dxdy \\
&= \int_{-\infty}^{0}\int_{-\infty}^{1} \frac{1}{4}e^{-|x|-|y|}dxdy \\
&= \frac{1}{4}\int_{-\infty}^{0} e^{-|y|}\left(\int_{-\infty}^{0} e^{x}dx + \int_{0}^{1} e^{-x}dx\right)dy \\
&= \frac{1}{4}\int_{-\infty}^{0} e^{-|y|}\left(|e^{x}|_{-\infty}^{0} + |-e^{-x}|_{0}^{1}\right)dy \\
&= \frac{1}{4}\int_{-\infty}^{0} e^{-|y|}\left((1-e^{-\infty}) - (e^{-1}-1)\right)dy \\
&= \frac{1}{4}\left(2-e^{-1}\right)\int_{-\infty}^{0} e^{y}dy \\
&= \frac{1}{4}\left(2-e^{-1}\right)|e^{y}|_{-\infty}^{0} \\
&= \frac{1}{4}\left(2-e^{-1}\right)
\end{aligned}
$$