Applied Mathematics for Electrical Engineering - 3130908

Unit-1: Numerical Solutions:

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1 Roots of Algebraic and Transcendental Equations

1.1 Introduction

Function $f(x) = 0$ is said to be algebraic if it is of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, where $a_n, a_{n-1}, \ldots, a_0$ are constants and n is a positive integer.

For example, $x^3 + 2x^2 - x + 4 = 0$.

Function $f(x) = 0$ is said to be transcendental if it contains function such as logrithmic, trignometric, exponential, etc.,

For example, $2x^3 - tan(x) + log(x^2) - xe^x = 0$.

Numerical methods are used to solve an equation when factorization methods are failed.

Here are some methods to solve the equation $f(x) = 0$,

(i) Bisection Method

(ii) Regula Falsi Method

(iii) Newton-Raphson Method

(iv) Secant Method

1.2 Bisection Method

Example 1 *Find the positive root of* $x^3 - 2x - 5 = 0$, *correct up to two decimal places.*

Solution:

We have $f(x) = x^3 - 2x - 5$

$f(1) = -6$ and $f(2) = -1$, $f(3) = 16$

Here $f(2) < 0$ and $f(3) > 0$

Therefore, the root lies between 2 and 3.

$x_1 = \frac{2+3}{2} = 2.5$, then $f(x_1) = f(2.5) = 5.625 > 0$.

Therefore, the root lies between 2.5 and 2. (Because, $f(2) < 0$ and $f(2.5) > 0$)

$x_2 = \frac{2+2.5}{2} = 2.25$, then $f(x_2) = f(2.25) = 1.8906 > 0$.

Therefore, the root lies between 2.25 and 2. (Because, $f(2) < 0$ and $f(2.25) > 0$)

$x_3 = \frac{2+2.25}{2} = 2.125$, then $f(x_3) = f(2.125) = 0.3457 > 0$.

Therefore, the root lies between 2.125 and 2. (Because, $f(2) < 0$ and $f(2.125) > 0$)

$x_4 = \frac{2+2.125}{2} = 2.0625$, then $f(x_4) = f(2.0625) = -0.3513 < 0$.

Hence, the root lies between 2.0625 and 2.125. (Because, $f(2.0625) < 0$ and $f(2.125) > 0$)

$x_5 = \frac{2.0625+2.125}{2} = 2.09375$, then $f(x_5) = f(2.09375) = -0.0089 < 0$.

Therefore, the root lies between 2.09375 and 2.125. (Because, $f(2.09375) < 0$ and $f(2.125) > 0$)

$x_6 = \frac{2.09375+2.125}{2} = 2.109375$, then $f(x_6) = f(2.109375) = 0.1668 > 0$.

Therefore, the root lies between 2.09375 and 2.109375. (Because, $f(2.09375) < 0$ and $f(2.109375) > 0$)

$x_7 = \frac{2.109375+2.09375}{2} = 2.10156$.

Since, x_6 and x_7 are same upto two decimal places.

Hence the require root is 2.109375

Example 2 *Find a root of the equation $x^3+x-1 = 0$, correct up to three decimal places.*

Solution:

We have $f(x) = x^3 + x - 1$

$f(0) = -1$ and $f(1) = 1$

Here $f(0) < 0$ and $f(1) > 0$

Therefore, the root lies between 0 and 1([0,1]).

$x_1 = \frac{0+1}{2} = 0.5$, then $f(x_1) = f(0.5) = -0.3750 < 0$.

Therefore, the root lies in the interval [0.5,1]. (Because, $f(2) < 0$ and $f(2.5) > 0$)

$x_2 = \frac{2+2.5}{2} = 2.25$, then Continue in this way, we get the following table values, Notice

No. Of Iteration	a	b	$x = \frac{a+b}{2}$	Sign of f
1	0	1	0.5	-ve
2	0.5	1	0.75	+ve
3	0.5	0.75	0.625	-ve
4	0.625	0.75	0.6875	+ve
5	0.625	0.6875	0.6563	-ve
6	0.6563	0.6875	0.6719	-ve
7	0.6719	0.6875	0.6797	-ve
8	0.6797	0.6875	0.6836	+ve
9	0.6797	0.6836	0.6817	-ve
10	0.6817	0.6836	0.56827	+ve
11	0.6817	0.56827	0.6822	

that, value of 10th and 11th iterations are same upto three decimal. Hence the require root is 0.6822

Example 3 *Find the positive root of the equation $2sinx - x = 0$, using bisection method in six stages.*

Solution:

We have $f(x) = 2sinx - x = 0$

$f(1) = 0.6829$ and $f(2) = -0.1814$

Here $f(2) < 0$ and $f(1) > 0$

Therefore, the root lies between 2 and 1([1,2]).

$x_1 = \frac{2+1}{2} = 1.5$, then $f(x_1) = f(1.5) = 0.4949 > 0$.

Thus the root lies between 1.5 and 2.

Continue in this way, we get the following table values, Hence the require root is 1.8907

No. Of Iteration	a	b	$x = \frac{a+b}{2}$	Sign of f
1	2	1	1.5	+ve
2	2	1.5	1.75	+ve
3	2	1.75	1.875	+ve
4	2	1.875	1.9375	-ve
5	1.9375	1.875	1.9063	-ve
6	1.9063	1.875	1.8907	+ve

1.3 Regula Falsi method

Method of false position or Regula falsi method is the oldest method for finding solutions of the equation $f(x) = 0$.

Working Rules for finding root of $f(x) = 0$

Let $f(x) = 0$

Srep 1. Choose two points a and b such that $f(a) < 0$ and $f(b) > 0$. Then

Srep 2. calculate first iteration using the formula,

$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$.

Srep 3. Calculate the value of $f(x_1)$. If $f(x_1) = 0$, then x_1 is a root of equation $f(x) = 0$. If $f(x_1) \neq 0$, then either $f(x_1) < 0$ or $f(x_1) > 0$. If $f(x_1) < 0$, then x_1 replaced with a

(i.e. the root lies in $[x_1 = a, b]$) and if $f(x_1) > 0$, then x_1 replaced with b (i.e. the root lies in $[a, x_1 = b]$). Find x_2 using the formula,

$x_2 = \frac{af(b)-bf(a)}{f(b)-f(a)}$.

Srep 4. Repeat this process until the root upto required decimals or steps.

Example 4 *Find the real root of the equation $xlog_{10}x = 1.2$, using by Regula-falsi method correct to four decimal places.*

Solution:

We have $f(x) = xlog_{10}x - 1.2 = 0$

$f(1) = -1.2 < 0$, $f(2) = -0.59794 < 0$, $f(3) = 0.23136 > 0$

Therefore, the root lies between 2 and 3($[2, 3]$).

Let $a = 2$ and $b = 3$. Using regula falsi method, $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{2f(3)-3f(2)}{f(3)-f(2)} = \frac{2(0.23136)-3(-0.59794)}{(0.23136)-(-0.59794)} = 2.72102$, then $f(x_1) = f(2.72102) = -0.01709 < 0$.

Thus the root lies between 2.72102 and 3([2.72102,3]).

Continue in this way, we get the following table values, Notice that 3rd and 4th iteration

No. Of Iteration	a	b	$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$	$f(x)$
1	2	3	2.72102	-0.01709
2	2.72102	3	2.74021	-0.00038
3	2.74021	3	2.74064	-0.00001
4	2.74064	3	2.74065	

are same upto four decimal places. Hence the require root is 2.74065

Example 5 *Find the real root of the equation $xe^x = 2$, using by Regula-falsi method correct to three decimal places.*

Solution:

We have $f(x) = xe^x - 2$

$f(0) = -2 < 0$ and $f(1) = 0.7183 > 0$

Therefore, the root lies between 0 and 1($[0, 1]$).

Let $a = 0$ and $b = 1$. Using regula falsi method, $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{0f(1)-1f(0)}{f(1)-f(0)} = \frac{0(0.7183)-1(-2)}{(0.7183)-(-2)} = 0.$, then $f(x_1) = f(2.72102) = -0.01709 < 0$.

Thus the root lies between 2.72102 and 3([2.72102,3]).

No. Of Iteration	a	b	$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$	$f(x)$
1	0	1	0.7358	-0.4644
2	0.73576	1	0.8395	-0.0563
3	0.8395	1	0.8512	-0.006171
4	0.8512	1	0.8525	-0.00067
5	0.85245	1	0.8526	-0.000072
6	0.8526	1	0.8526	-0.0000078

Continue in this way, we get the following table values, Notice that 5th and 6th iteration are same upto three decimal places. Hence the require root is 0.8526

1.4 Secant Method

Working Rules for finding root of $f(x) = 0$

Let $f(x) = 0$

Srep 1. Choose two points x_0 and x_1 such that $f(x_0) < 0$ and $f(x_1) > 0$. Then

Srep 2. calculate the x_2 iteration using the formula,

$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$.

Srep 3. calculate the next iteration x_3 using the formula. $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$.

Srep 4. Repeat this process until the root upto required decimals or steps.

Example 6 *Solve the equation* $e^{-x} - tanx = 0$, *correct to three decimal places by Secant method, starting from* $x_0 = 1$, $x_1 = 0.7$.

Solution:

We have $f(x) = e^{-x} - tanx$

Given that $x_0 = 1$, $x_1 = 0.7$. By the secant method

$$\begin{aligned} x_2 =& \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\ =& \frac{(1)f(0.7) - (0.7)f(1)}{f(0.7) - f(1)} \\ =& \frac{(1)(-0.3457) - (0.7)(-1.1895)}{(-0.3457) - (-1.1895)} \\ =& 0.5771 \end{aligned}$$

Next iteration x_3 by,

$$\begin{aligned} x_3 =& \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \\ =& \frac{(0.7)f(0.5771) - (0.5771)f(0.7)}{f(0.5771) - f(0.7)} \\ =& \frac{(0.7)(-0.0895) - (0.5771)(-0.3457)}{(-0.0895) - (-0.3457)} \\ =& 0.5342 \end{aligned}$$

Next iteration x_4 by,

$$\begin{aligned} x_4 =& \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \\ =& \frac{(0.5771)f(0.5342) - (0.5342)f(0.5771)}{f(0.5342) - f(0.5771)} \\ =& \frac{(0.5771)(-0.0054) - (0.5771)(-0.0895)}{(-0.0054) - (-0.0895)} \\ =& 0.5318 \end{aligned}$$

Next iteration x_5 by,

$$\begin{aligned} x_5 =& \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} \\ =& \frac{(0.5342)f(0.5318) - (0.5318)f(0.5342)}{f(0.5318) - f(0.5342)} \\ =& \frac{(0.5342)(-0.0008) - (0.5318)(-0.0054)}{(-0.0008) - (-0.0054)} \\ =& 0.5318 \end{aligned}$$

From x_4 and x_5 the required root upto three decimal is 0.5318.

Example 7 *Find a real root of the equation $x - 2sinx = 0$, correct to six decimal places by Secant method, starting from $x_0 = 2$, $x_1 = 1.9$.*

Solution:

We have $f(x) = x - 2sinx = 0$

Given that $x_0 = 2$, $x_1 = 1.9$. By the secant method

$$\begin{aligned} x_2 =& \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\ =& \frac{(2)f(1.9) - (1.9)f(2)}{f(1.9) - f(2)} \\ =& \frac{(2)(0.0073998) - (1.9)(0.18140515)}{(0.0073998) - (0.18140515)} \\ =& 1.895747 \end{aligned}$$

Next iteration x_3 by,

$$\begin{aligned} x_3 =& \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \\ =& \frac{(1.9)f(1.895747) - (1.895747)f(1.9)}{f(1.895747) - f(1.9)} \\ =& \frac{(1.9)(0.00082866) - (1.896)(0.0073998)}{(0.00082866) - (0.0073998)} \\ =& 1.895495 \end{aligned}$$

Next iteration x_4 by,

$$\begin{aligned} x_4 =& \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \\ =& \frac{(1.895747)f(1.895495) - (1.895495)f(1.895747)}{f(1.895495) - f(1.895747)} \\ =& \frac{(1.896)(0.00000939) - (1.8955)(0.00082866)}{(0.00000939) - (0.00082866)} \\ =& 1.895494 \end{aligned}$$

Next iteration x_5 by,

$$\begin{aligned} x_5 =& \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} \\ =& \frac{(1.895495)f(1.895494) - (1.895494)f(1.895495)}{f(1.895494) - f(1.895495)} \\ =& \frac{(1.895495)(0.0000903) - (1.895494)(0.00000939)}{(0.0000903) - (0.00000939)} \\ =& 1.895494 \end{aligned}$$

From x_4 and x_5 the required root upto three decimal is 1.895494.

1.5 Newton-Raphson method

Newton-Raphson method is used to find a root for the equation $f(x) = 0$, where $f(x)$ is continuously differentiable. This method is very fast in convergence.

Working Rules for finding root of $f(x) = 0$

Let $f(x) = 0$

Srep 1. For the initial guess is given then say it x_0, otherwise find from the equation $x_0 = \frac{a+b}{2}$, where $x = a$ and $x = b$ such that $f(a) < 0$ and $f(b) > 0$.

Srep 2. Find first derivative $f'(x)$ from $f(x) = 0$.

Srep 3. Apply Newton-Raphson formula

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ (n=0,1,2,...)

to find values of $x_1, x_2, \ldots$ of approximate roots. Stop when the root upto desired accuracy is obtained.

Example 8 *Find a real smallest root of the equation $sinx = e^{-x}$, correct to four decimal places by using Newton-Raphson method, starting from $x_0 = 0.6$.*

Solution:

We have $f(x) = sinx - e^{-x} = 0$. then $f'(x) = cosx + e^{-x} = 0$

From the equation of Newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n} \, (n = 0, 1, 2, ...) \tag{1}$$

For first iteration let n=0, and put values given that $x_0 = 0.6$ in equation (2) we get,

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.6 - \frac{f(0.6)}{f'(0.6)} = 0.6 - 0.01152 = 0.58848$

For n=1 in (2), we get

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.58848 - \frac{f(0.58848)}{f'(0.58848)} = 0.58848 + 0.00005 = 0.58853$

For n=2 in (2), we get

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.58853 - \frac{f(0.58853)}{f'(0.58853)} = 0.58853 + 0.0000072 = 0.58853$

Hence, the required root is 0.58853 upto four decimal places.

Example 9 *Find a root of the equation $x^4 - x^3 + 10x + 7 = 0$, correct to three decimal places by using Newton-Raphson method, given that $a = -2$ and $b = -1$.*

Solution:

We have $f(x) = x^4 - x^3 + 10x + 7$. then $f'(x) = 4x^3 - 3x^2 + 10$

$x_0 = \frac{a+b}{2} = \frac{-1-2}{2} = -1.5$

From the equation of Newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n} \, (n = 0, 1, 2, ...) \tag{2}$$

For first iteration let n=0, and put values given that $x_0 = -1.5$ in equation (2) we get,

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1.5 - \frac{f(-1.5)}{f'(-1.5)} = -1.5 - \frac{0.4375}{-10.25} = -1.5 - (-0.04268) = -1.45732$

For n=1 in (2), we get

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.45732 - \frac{f(-1.45732)}{f'(-1.45732)} = -1.45732 - \frac{0.032252}{-8.75136} = -1.45732 - (-0.00366853) = -1.453632$

For n=2 in (2), we get

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -1.453632 - \frac{f(-1.453632)}{f'(-1.453632)} = -1.453632 - \frac{0.000232}{-8.6255} = -1.453632 - (-0.0000269) = -1.453605$

Hence, the required root is -1.453605 upto three decimal places.

1.5.1 Iterative formula to finding $\sqrt{N}$

Suppose that $x = \sqrt{N}$. Then $x^2 = N$. Let $f(x) = x^2 - N$ then $f'(x) = 2x$.

From the Newton-Raphson formula,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \, (n = 0, 1, 2, ...) \\ \Rightarrow x_{n+1} &= x_n - \frac{(x_n)^2 - N}{2x_n} \\ \Rightarrow x_{n+1} &= \frac{2{x_n}^2}{2x_n} - \frac{{x_n}^2}{2x_n} + \frac{N}{2x_n} \\ \Rightarrow x_{n+1} &= \frac{x_n}{2} + \frac{N}{2x_n} \\ \Rightarrow x_{n+1} &= \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \end{aligned} \tag{3}$$

which is an iterative formula to obtaining $\sqrt{N}$, where $N > 0$.

Example 10 *Find approximate root of* $\sqrt{28}$*. correct to four decimal places.*

Solution:

Let N=28 and $x_0 = 5$. (Because $\sqrt{28} \approx 5$). For n=0 in (3), we get

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right) = \frac{1}{2}\left(5 + \frac{28}{5}\right) = 5.3$$

For n=1 in (3), we get

$$x_2 = \frac{1}{2}\left(x_1 + \frac{28}{x_1}\right) = \frac{1}{2}\left(5.3 + \frac{28}{5.3}\right) = 5.29151$$

.

For n=2 in (3), we get

$$x_3 = \frac{1}{2}\left(x_2 + \frac{28}{x_2}\right) = \frac{1}{2}\left(5.29151 + \frac{28}{5.29151}\right) = 5.29150$$

.

Hence $\sqrt{28} = 5.29150$

Example 11 *Find approximate root of* $\sqrt{65}$.

Solution:

Let N=65 and $x_0 = 8$. (Because $\sqrt{65} \approx 8$). For n=0 in (3), we get

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right) = \frac{1}{2}\left(8 + \frac{65}{8}\right) = 8.0625$$

For n=1 in (3), we get

$$x_2 = \frac{1}{2}\left(x_1 + \frac{65}{x_1}\right) = \frac{1}{2}\left(8.0625 + \frac{65}{8.0625}\right) = 8.06226$$

.

For n=2 in (3), we get

$$x_3 = \frac{1}{2}\left(x_2 + \frac{65}{x_2}\right) = \frac{1}{2}\left(8.06226 + \frac{65}{8.06226}\right) = 8.06226$$

.

Hence $\sqrt{65} = 8.06226$

1.6 Fixed Point Iteration or Iteration method

Working Rules for finding root of $f(x) = 0$

Let $f(x) = 0$

Srep 1. For the initial guess is given then say it x_0, otherwise find from the equation

$x_0 = \frac{a+b}{2}$, where $x = a$ and $x = b$ such that $f(a) < 0$ and $f(b) > 0$.

Srep 2. Make x subject as $x = \phi(x)$ such that $|\phi'(x)| < 1$.

Srep 3. Finding roots $x_1, x_2, \ldots$ from the formula,

$$x_{n+1} = \phi(x_n)\,(n = 0, 1, 2, ...)$$

Srep 4. Repeat this process until the root upto required decimals or steps.

Example 12 *Find a root of $x^4 - x - 10 = 0$, by using Iteration method correct upto four decimal places.*

Solution:

We have $f(x) = x^4 - x - 10 < 0 = 0$. then $f(0) = -10 < 0$, $f(1) = -10 < 0$, $f(2) = 4 > 0$
Hence, the root lies between 1 and 2.(i.e. [1,2])
Rewrite equation $x^4 - x - 10 < 0 = 0$ in terms of of x as,
Case-I: $x = \frac{10}{x^3-1} = \phi(x)$
Case-II:$x = (x + 10)^{1/4} = \phi(x)$
Case-III:$x = \frac{(x+10)^{1/2}}{x} = \phi(x)$
Notice from the above equations, for the second choice $|\phi'(x)| = 0.0400 < 1$ at $x_0 = 1.5$, which is the smallest among to all.
Therefore, formula becomes,
$x_{n+1} = (x_n + 10)^{1/4} = \phi(x_n)$ For n=0 we get,

$$\begin{aligned} x_1 =&(x_0 + 10)^{1/4} \\ =&(1.5 + 10)^{1/4} \\ =&(11.5)^{1/4} \\ =&1.841512 \end{aligned}$$

For n=1 we get,

$$\begin{aligned} x_2 =&(x_1 + 10)^{1/4} \\ =&(1.841512 + 10)^{1/4} \\ =&(11.841512)^{1/4} \\ =&1.855034 \end{aligned}$$

For n=2 we get,

$$
\begin{aligned}
x_3 =& (x_2 + 10)^{1/4} \\
=& (1.855034 + 10)^{1/4} \\
=& (11.855034)^{1/4} \\
=& 1.85556
\end{aligned}
$$

For n=3 we get,

$$
\begin{aligned}
x_4 =& (x_3 + 10)^{1/4} \\
=& (1.85556 + 10)^{1/4} \\
=& (11.85556)^{1/4} \\
=& 1.85556
\end{aligned}
$$