Applied Mathematics for Electrical Engineering - 3130908

Unit-2: Interpolation

Contents

1 Introduction

Interpolation is the process of reading between the lines of a table. It is the process of computing intermediate values of a function from a given set of tabular values of the function. Extrapolation is used to denote the process of finding the values outside the given interval.

In the interpolation process, the given set of tabular values are used to find an expression for $f(x)$ and then using it to find its required value for a given value of x. But, it is difficult to find an exact form of $f(x)$ using limited values in the table. Hence, $f(x)$ is replaced by another function $\phi(x)$, which matches with $f(x)$ at the discrete values in the table. This function $\phi(x)$ is known as the interpolating function. When the interpolating function is a polynomial function, the process is known as polynomial interpolation.

2 Finite Differences

Let the function $y = f(x)$ be tabulated for the equally spaced values $y_0 = f(x_0)$, $y_1 = f(x_0 + h)$, $y_2 = f(x_0 + 2h)$, ..., $y_n = f(x_0 + nh)$, as

X	x_0	$x_0 + h$	$x_0 + 2h$	...	$x_0 + nh$	...
$y = f(x)$	y_0	y_1	y_2	...	y_n	...

2.1 Forward Differences

If y_0 , y_1 , y_2 , ... , y_n denote a set of values of y then Δy_0, Δy_1, ..., Δy_{n-1} are called first forward differences and defined by,

$$\begin{aligned}
\Delta y_0 &= y_1 - y_0 \\
\Delta y_1 &= y_2 - y_1 \\
\vdots &= \vdots \\
\Delta y_{n-1} &= y_n - y_{n-1}
\end{aligned}$$

where Δ is called the forward difference operator.

Similarly, second, third, forth forward differences, etc can be defined as mentioned bellow,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
$x_1 = x_0 + h$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
$x_2 = x_0 + 2h$	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
$x_3 = x_0 + 3h$	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
		$\Delta y_3 = y_4 - y_3$			
$x_4 = x_0 + 4h$	y_4				

2.2 Backward Differences

If y_0 , y_1 , y_2 , ... , y_n denote a set of values of y then ∇y_1, ∇y_2, ..., $nablay_n$ are called first backward differences and defined by,

$$\nabla y_1 = y_1 - y_0, \quad \nabla y_2 = y_2 - y_1, \quad \ldots, \quad \nabla y_n = y_n - y_{n-1}$$

where ∇ is called the backward difference operator.
Similarly, second, third, forth backward differences, etc can be defined as mentioned bellow,

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		$\nabla y_1 = y_1 - y_0$			
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$		
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$	
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$		$\nabla^4 y_4 = \nabla^3 y_4 - \nabla^3 y_3$
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$	
x_3	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$		
		$\nabla y_4 = y_4 - y_3$			
x_4	y_4				

2.3 Central Differences

If y_0 , y_1 , y_2 , ... , y_n denote a set of values of y then $\delta y_{\frac{1}{2}}$, $\delta y_{\frac{3}{2}}$, ..., $\delta y_{n-\frac{1}{2}}$ are called first central differences and defined by,

$$\begin{aligned} \delta y_{\frac{1}{2}} &= y_1 - y_0 \\ \delta y_{\frac{3}{2}} &= y_2 - y_1 \\ \vdots &= \vdots \\ \delta y_{n-\frac{1}{2}} &= y_n - y_{n-1} \end{aligned}$$

where δ is called the central difference operator.

Similarly, second, third, forth central differences, etc can be defined as mentioned bellow,

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{\frac{1}{2}} = y_1 - y_0$			
x_1	y_1		$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$		
		$\delta y_{\frac{3}{2}} = y_2 - y_1$		$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1$	
x_2	y_2		$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}}$		$\delta^4 y_2 = \delta^3 y_{\frac{5}{2}} - \delta^3 y_{\frac{3}{2}}$
		$\delta y_{\frac{5}{2}} = y_3 - y_2$		$\delta^3 y_{\frac{5}{2}} = \delta^2 y_3 - \delta^2 y_2$	
x_3	y_3		$\delta^2 y_3 = \delta y_{\frac{7}{2}} - \delta y_{\frac{5}{2}}$		
		$\delta y_{\frac{7}{2}} = y_4 - y_3$			
x_4	y_4				

3 Newton's Forward interpolation formula

Let the function $y = f(x)$ take the values $y_1, y_2, ...$ corresponding to the values $x_0, x_1, ...$of x. Suppose that it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number. Then the Newton's forward interpolation formula defined as,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... \quad (1)$$

where

$$p = \frac{x - x_0}{h}$$

Example 1 *Compute cosh(0.56) using Newton's forward difference formula from the following data,*

X	0.5	0.6	0.7	0.8
$y = f(x)$	1.127626	1.185465	1.255169	1.337435

Solution:

Let $x = 0.56$, $x_0 = 0.5$, $h = 0.1$ then $p = \frac{x-x_0}{h} = \frac{0.56-0.5}{0.1} = 0.6$

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.5	1.127626			
		$\Delta y_0 = 0.057839$		
0.6	1.185465		$\Delta^2 y_0 = 0.011865$	
		$\Delta y_1 = 0.069704$		$\Delta^3 y_0 = 0.000697$
0.7	1.2551692		$\Delta^2 y_1 = 0.012562$	
		$\Delta y_2 = 0.082266$		
0.8	1.337435			

From the Newton's forward difference formula (1) ,

$$
\begin{aligned}
y_p = f(x) =& y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... \\
Cos(0.56) =& 1.127626 + 0.6(0.057839) + \frac{0.6(0.6-1)}{2!}(0.011865) + \frac{0.6(0.6-1)(0.6-2)}{3!}(0.000697) \\
=& 1.127626 + 0.034703 - 0.001424 + 0.000039 \\
=& 1.160944
\end{aligned}
$$

Example 2 *Evaluate sin*$(52°)$ *using Newton's forward interpolation formula from the following data ;*

$\theta°$	45°	50°	55°	60°
$sin\theta°$	0.7071	0.7660	0.8192	0.8660

Solution:

Let $x = 52°$, $x_0 = 45°$, $h = 5°$ then $p = \frac{x-x_0}{h} = \frac{52°-45°}{5°} = 1.4°$

Difference table

$x = \theta°$	$y = sin\theta°$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		$\Delta y_0 = 0.0589$		
50°	0.7660		$\Delta^2 y_0 = -0.0057$	
		$\Delta y_1 = 0.0532$		$\Delta^3 y_0 = -0.0007$
55°	0.8192		$\Delta^2 y_1 = -0.0064$	
		$\Delta y_2 = 0.0468$		
60°	0.8660			

From the Newton's forward difference formula (1) ,

$$\begin{aligned} y_p = f(x) =& y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... \\ Sin(52°) =& 0.7071 + 1.4(0.0589) + \frac{1.4(1.4-1)}{2!}(0 - 0.0057) + \frac{1.4(1.4-1)(1.4-2)}{3!}(-0.0007) \\ =& 0.7071 + 0.0825 - 0.00146 + 0.00004 \\ =& 0.7880 \end{aligned}$$