Fundamental Mathematics Concepts

CC-104

Unit-2: Matrices and Determinants

Contents

1 Matrices

1.1 Definitions

Definition 1 (Matrix) *A system of a rectangle formation along m rows and n columns of mn numbers and bounded by ( ) or [ ] is called m by n MATRIX, it is denoted by $m \times n$ matrix. Matrix is denoted by capital letters.*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \quad (1)$$

Equation (1) is a matrix of order mn, with m rows and n columns and each members from matrix is called element of the matrix.
The matrix A is also denoted as,
$A = [a_{ij}]_{m\times n}$
where i^{th} letter indicate rows of the matrix and j^{th} letter indicate columns of the matrix.

Definition 2 (Row) *A matrix with single row and multiple column is called Row matrix.*

For example: $\begin{bmatrix} 2 & 5 & 1 & 6 \end{bmatrix}$

Definition 3 (Column) *A matrix with single column and multiple row is called Row matrix.*

For example: $\begin{bmatrix} 2 \\ 1 \\ 8 \\ 0 \end{bmatrix}$

Definition 4 (Square) *A matrix having rows and column are equal then it is called square matrix.*

For example: $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 2 & 1 & 0 \\ 9 & 1 & 3 \\ 7 & 5 & 4 \end{bmatrix}_{3\times 3}$, $\begin{bmatrix} 6 & 2 & 1 & 0 \\ -1 & 9 & 1 & 3 \\ 2 & 7 & 5 & 4 \\ 1 & 0 & 1 & 4 \end{bmatrix}_{4\times 4}$

Definition 5 (Zero) *A matrix having all the entries are zero is called zero or null matrix.*

For example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

Definition 6 (Diagonal) *A matrix with all entries are zero except its diagonal is called diagonal matrix.*

For example: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$, $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}_{4\times 4}$

Definition 7 (Scalar) *A diagonal matrix having equal elements is called scalar matrix.*

For example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$, $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}_{4\times 4}$

Definition 8 (Identity) *A diagonal square matrix having all the entries are 1 is called identity or unit matrix.*

For example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

Definition 9 (Symmetric) *A square matrix $A = [a_{ij}]$ with $a_{ij} = a_{ji}$ for all i and j is called symmetric matrix.*

For example: $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 1 & 5 & 8 \\ 5 & 1 & 0 \\ 8 & 0 & 2 \end{bmatrix}_{3\times 3}$

Definition 10 (Skew-symmetri) *A square matrix $A = [a_{ij}]$ with $a_{ij} = -a_{ji}$ for all i and j is called skew-symmetric matrix.*

For example: $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 0 & 5 & 8 \\ -5 & 0 & 0 \\ -8 & 0 & 0 \end{bmatrix}_{3\times 3}$

Definition 11 (Transpose Matrix) *The matrix obtained from any given matrix A by interchanging rows and columns is called the transpose of A and it is denoted as A'.*

For example: IF $A = \begin{bmatrix} 2 & 1 & 0 \\ 9 & 1 & 3 \\ 7 & 5 & 4 \end{bmatrix}$ then its transpose matrix $A' = \begin{bmatrix} 2 & 9 & 7 \\ 1 & 1 & 5 \\ 0 & 3 & 4 \end{bmatrix}$

Definition 12 (Addition of Matrices) *Addition of two matrices A and B (order A and B are same) is defined as the matrix which is the sum of the corresponding elements of A and B.*

Definition 13 (Subtraction of Matrices) *Subtraction of two matrices A and B (order A and B are same) is defined as the matrix is obtained by subtracting the elements of B from the corresponding elements of A.*

Definition 14 (Multiplication of a matrix by a scalar) *If matrix A is multiply a scalar k then its each elements is multiplied by corresponding element by k.*

Example 1 *Find the values of a, b, c, x, y, z from the following relation.*

$$\begin{bmatrix} a+1 & b+2 & 3+z \\ -5 & c-7 & 0 \\ x+6 & y+4 & 1 \end{bmatrix} = \begin{bmatrix} 2a+5 & 7 & 2z-5 \\ -5 & 0 & x \\ 6 & 5 & 1 \end{bmatrix}$$

Solution:

From the definition of equality of two matrices,

$$a + 1 = 2a + 5 \implies a = -4$$

$$b + 2 = 7 \implies b = 5$$

$$3 + z = 2z - 5 \implies z = 8$$

$$c - 7 = 0 \implies c = 7$$

$$0 = x \implies x = 0$$

$$y + 4 = 5 \implies y = 1$$

hence, $a = -4, b = 5, c = 7, x = 0, y = 1$ and $z = 8$.

Example 2 *Find* x, y, z, w *if* $\begin{bmatrix} x+y & 2z+w \\ x-y & z-w \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$

Solution:

From the definition of equality of two matrices,

$$x + y = 3$$
$$2z + w = 5$$
$$x - y = 1$$
$$z - w = 4$$

Solving these equations we get, $x = 2, y = 1, z = 3, w = -1$.

Example 3 *Find* $A + B$ *if* $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{bmatrix}$

Solution: $A + B = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 5 \\ -3 & 6 & 2 \end{bmatrix}$

Example 4 *Find* $2A + 5B$ *if* $A = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}, B = \begin{bmatrix} 4 & -3 & -6 \\ 3 & 7 & -8 \end{bmatrix}$

Solution:

The sum $2A + 5B$ is not possible because, dimensions of A and B are different.

Example 5 *Find* $3A - 4B$ *if* $A = \begin{bmatrix} 3 & -4 & 6 \\ 5 & 1 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Solution:

$3A = \begin{bmatrix} 9 & -12 & 18 \\ 15 & 3 & 21 \end{bmatrix}$ and $4B = \begin{bmatrix} 4 & 0 & 4 \\ 8 & 0 & 12 \end{bmatrix}$

$\therefore 3A - 4B = \begin{bmatrix} 9 & -12 & 18 \\ 15 & 3 & 21 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 4 \\ 8 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 5 & -12 & 14 \\ 7 & 3 & 9 \end{bmatrix}$

Definition 15 (Multiplication of Matrices) *If two matrices* $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix}$

and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix}$ *then matrix multiplication of* A *and* B *is denoted by* $A \times B$ *or* AB

and defined as,

$$AB = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} = \begin{bmatrix} a_1b_1 + a_2b_3 + a_3b_5 & a_1b_2 + a_2b_4 + a_3b_6 \\ a_4b_1 + a_5b_3 + a_6b_5 & a_4b_2 + a_5b_4 + a_6b_6 \\ a_7b_1 + a_8b_3 + a_9b_5 & a_7b_2 + a_8b_4 + a_9b_6 \\ a_{10}b_1 + a_{11}b_3 + a_{12}b_5 & a_{10}b_2 + a_{11}b_4 + a_{12}b_6 \end{bmatrix}$$

Note: Two matrices can be multiplied if the number of columns in first matrix is equal to number of rows in the second matrix.

In general, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times l}$ then their product is defined as $C = [c_{ij}]_{m\times l}$.

Properties of Multiplication of two matrices

1. Post and pre multiplication $IA = AI = A$
2. $(AB)C = A(BC)$ (Associative)
3. $A(B + C) = AB + AC$
4. Powers of matrix $A^3 = A^2 \cdot A$ $A^4 = A^3 \cdot A$ etc.
5. $AB \neq BA$
6. Orthogonal $AA^T = A^TA = I$
7. Any $n \times n$ matrix A holds $A^n = 0$ then A is called NILPOTENT.
8. Any $n \times n$ matrix A holds $A^2 = A$ then A is called IDEMPOTENT.
9. Any $n \times n$ matrix A holds $A^2 = I$ then A is called INVOLUNTARY.

Example 6 *Find* $\begin{bmatrix} 8 & -4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

Solution:

$$\begin{aligned} \begin{bmatrix} 8 & -4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} &= (8)(3) + (-4)(2) + (5)(-1) \\ &= 24 - 8 - 5 \\ &= 11 \end{aligned}$$

Example 7 *Find AB if* $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$

Solution: Here, matrix A is 2×2 and matrix B is of 2×3, then matrix multiplication is possible between A and B.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix} \\ &= \begin{bmatrix} (1)(2)+(3)(3) & (1)(0)+(3)(-2) & (1)(-4)+(3)(6) \\ (2)(2)+(-1)(3) & (2)(0)+(-1)(-2) & (2)(-4)+(-1)(6) \end{bmatrix} \\ &= \begin{bmatrix} 11 & -6 & 14 \\ 1 & 2 & -14 \end{bmatrix} \end{aligned}$$

Example 8 *Find AB and BA if* $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$

Solution: Here, matrix A is 3×2 and matrix B is of 2×3, then matrix multiplication is possible between A and B. AB will be3×3 and BA will be2×2

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2-3 & -4-4 & -10+0 \\ 1+0 & -2+0 & -5+0 \\ -3+12 & 6+16 & 15+0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix} \end{aligned}$$

$$BA = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 2-2+15 & -1+0-20 \\ 6+4+00 & -3+0+0 \end{bmatrix}$$
$$= \begin{bmatrix} 15 & -21 \\ 10 & -3 \end{bmatrix}$$

Example 9 *If* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$ *then prove that* $(AB)' = B'A'$.

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 20 & 4 \end{bmatrix}$$
$$\therefore\ (AB)' = \begin{bmatrix} 8 & 20 \\ 2 & 4 \end{bmatrix}$$
$$B' = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$
$$\therefore\ B'A' = \begin{bmatrix} 8 & 20 \\ 2 & 4 \end{bmatrix}$$

Hence, $(AB)' = B'A'$.

Example 10 *Find* AA^T *and* A^TA *if* $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$.

Solution:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}, \; A^T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$$

$$\therefore \; AA^T = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 26 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{bmatrix}$$

Definition 16 (Determinant) *The symbol* $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$ *where* a_1, a_2, b_1, b_2 *are numbers is called a determinant of second order.*

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ *is called a determinant of third order. and its value is defined as,*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Properties of determinants

1. The determinant's value is same, if the corresponding rows and columns are interchanged.
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. The value of the determinants become minus(-) if any two rows or columns are interchanged. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

3. If any constat $k \in R$ multiply by determinants then it is multiply by each elements of a row or column.

4. $\begin{vmatrix} a_1 + d_1 & b_1 + d_2 & c_1 + d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

5. If elements of any two rows or column are same then value of determinants is zero.

6. The value of determinants is not changed if any rows or column added or subtracted to another with any constant k.

Definition 17 (Co-factors) *If* $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ *is any determinants then co-factors of* D *is determine by following way, The cofactor of*

$$a_1 is A_1 = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$b_1 is B_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$c_1 is C_1 = (-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Cramer's Rule

Let two equations in the two unknowns x and y.

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

System has unique solution if and only if $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0.$

The solution of the system is obtaining as,

$$x = \frac{D_1}{D} \text{ and } y = \frac{D_2}{D}$$

where $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Let three equations in the three unknowns x, y and z.

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

System has unique solution if and only if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$

The solution of the system is obtaining as,

$$x = \frac{D_1}{D},\ y = \frac{D_2}{D} \text{ and } z = \frac{D_3}{D}$$

where $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Example 11 *Solve:* $2x - 3y = 7$ *and* $3x + 5y = 1$.

Solution:

Here, $D = \begin{vmatrix} 2 & -3 \\ 3 & 5 \end{vmatrix} = 19 \neq 0$

So, the system has a unique solution. We find D_1 and D_2.

$$D_1 = \begin{vmatrix} 7 & -3 \\ 1 & 5 \end{vmatrix} = 38,\ D_2 = \begin{vmatrix} 2 & 7 \\ 3 & 1 \end{vmatrix} = -19$$

Thus,

$$x = \frac{D_1}{D} = \frac{38}{19} = 2$$

$$y = \frac{D_2}{D} = \frac{-19}{19} = -1$$

Example 12 *Solve:* $2x + y - z = 3$, $x + y + z = 1$ *and* $x - 2y - 3z = 4$.

Solution:

Here, $D = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix} = 5 \neq 0$. So, the system has a unique solution.

$$D_1 = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{vmatrix} = 10,\ D_2 = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix} = -5,\ D_3 = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix} = 0$$

Thus,

$$x = \frac{D_1}{D} = \frac{10}{5} = 2$$
$$y = \frac{D_2}{D} = \frac{-5}{5} = -1$$
$$z = \frac{D_3}{D} = \frac{0}{5} = 0$$

Row and Column operations

$R_{13}(3)$= Add first row to the third row by multiply with 3.

$R_{32}(-4)$= Add third row to the second row by multiply with -4.

$C_{13}(3)$= Add first column to the third column by multiply with 3.

$C_{32}(-4)$= Add third column to the second column by multiply with -4.

Example 13 *Show that* $\begin{vmatrix} y+z & x-y & x \\ z+x & y-z & y \\ x+y & z-x & z \end{vmatrix} = 3xyz - x^3 - y^3 - z^3$ *using theorem of determinants.*

Solution:

$$\begin{vmatrix} y+z & x-y & x \\ z+x & y-z & y \\ x+y & z-x & z \end{vmatrix}$$
$$= \begin{vmatrix} x+y+z & -y & x \\ x+y+z & -z & y \\ x+y+z & -x & z \end{vmatrix} \quad (\textit{Applying}\, C_{31}(1),\ C_{32}(-1))$$

$$= -(x+y+z)\begin{vmatrix} 1 & y & x \\ 1 & z & y \\ 1 & x & z \end{vmatrix}$$

$$= -(x+y+z)[1(z^2 - xy) - y(z-y) + x(x-z)]$$

$$= -(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= -(x^3 + y^3 + z^3 - 3xyz)$$

$$= 3xyz - x^3 - y^3 - z^3$$

Definition 18 (Adjoint of a matrix) *Adjoint of a square matrix A is denoted as $Adj(A)$ and defined by,*

$$Adj(A) = \textit{Transpose matrix a cofactor matrix of } A = C^T$$

where C=Co-factor matrix of A.

NOTE: For two by two matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ then $Adj(A) = \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}$

For example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $Adj(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Definition 19 (Invertible matrix) *If determinants of matrix A is non-zero i.e. $|A| \neq 0$ then Matrix A is called invertible.*

For example: $If A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ then its determinant $|A| = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2 \neq 0.$
Hence, A is invertible.

Definition 20 (Inverse of a matrix) *Inverse of matrix A is denoted as A^{-1} and defined by,*

$$A^{-1} = \frac{Adj(A)}{|A|}$$

Example 14 *Find adjoint of* $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & -3 \\ 2 & 1 & 1 \end{bmatrix}$

Solution:

Cofactor of 2 $= \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 1 + 3 = 4$

Cofactor of 0 $= -\begin{vmatrix} 0 & -3 \\ 2 & 1 \end{vmatrix} = -(0 + 6) = -6$

Cofactor of 2 $= \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = 0 - 2 = -2$

Cofactor of 0 $= -\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -(0 - 2) = 2$

Cofactor of 1 $= \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 2 - 4 = -2$

Cofactor of -3 $= -\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = -(2 - 0) = -2$

Cofactor of 2 $= \begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} = 0 - 2 = -2$

Cofactor of 1 $= -\begin{vmatrix} 2 & 2 \\ 0 & -3 \end{vmatrix} = -(-6 - 0) = 6$

Cofactor of 1 $= \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 - 0 = 2$

Cofactor matrix $C = \begin{bmatrix} 4 & -6 & -2 \\ 2 & -2 & -2 \\ -2 & 6 & 2 \end{bmatrix}$

$\therefore\ Adj(A) = \begin{bmatrix} 4 & 2 & -2 \\ -6 & -2 & 6 \\ -2 & -2 & 2 \end{bmatrix}$

Example 15 *Obtain the inverse of the matrix* $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Solution:

Here, $|A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix} = -8 \neq 0$ So, matrix is invertible.

The cofactor matrix of $A = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}$ $\quad \therefore \ Adj(A) = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$

$$\therefore \ A^{-1} = \frac{Adj(A)}{|A|} = \frac{\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}}{-8}$$

$$= \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$$

Solution of linear equations (Matrix Inversion Method)

Let three equations in the three unknowns x, y and z.

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

This system have unique solution if determinants of the matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is non-zero (i.e. $|A| \neq 0$)

The above system of equations can be expressed in matrix form as,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

i.e. $AX = B$

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

then $x_1 = p_1$, $x_2 = p_2$, $x_3 = p_3$.

Example 16 *Using matrix inversion method find the solution of the following system,*
$2x - 3y + 5 = 0\; 3x + y - 9 = 0$

Solution:

Write the given system into matrix form as, $\begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \end{bmatrix}$

. Here $A = \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -5 \\ 9 \end{bmatrix}$

$|A| = 11 \neq 0$, A^{-1} exists.

$\therefore$ the system of equations has a unique solution.

$$Adj(A) = \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$$

$$\therefore\; A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{11} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 9 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$x = 2$, $y = 3$

Example 17 *Using matrix inversion method find the solution of the following system,*

$x + 2y + 3z = 6$

$2x + 4y + z = 7$

$3x + 2y + 9z = 14$

Solution:

Write the given system into matrix form as, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 14 \end{bmatrix}$

. Here $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 7 \\ 14 \end{bmatrix}$

$|A| = -20 \neq 0$, A^{-1} exists.

$\therefore$ the system of equations has a unique solution.

$$Cofactor matrix C = \begin{bmatrix} 34 & -15 & -8 \\ -12 & 0 & 4 \\ -10 & 5 & 0 \end{bmatrix}$$

$$Adj(A) = \begin{bmatrix} 34 & -12 & -10 \\ -15 & 0 & 5 \\ -8 & 4 & 0 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{-20} \begin{bmatrix} 34 & -12 & -10 \\ -15 & 0 & 5 \\ -8 & 4 & 0 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-20} \begin{bmatrix} 34 & -12 & -10 \\ -15 & 0 & 5 \\ -8 & 4 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 14 \end{bmatrix} = \frac{1}{-20} \begin{bmatrix} -20 \\ -20 \\ -20 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$x = 1$, $y = 1$ and $z = 1$

Example 18 *Using matrix inversion method find the solution of the following system,*
$3x + 5y - 8z = 6$ $5x - 2y - 3z = -1$ $2x - 6y + 4z = 1$

Solution:

Write the given system into matrix form as, $\begin{bmatrix} 3 & 5 & -8 \\ 5 & -2 & -3 \\ 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix}$

. Here $A = \begin{bmatrix} 3 & 5 & -8 \\ 5 & -2 & -3 \\ 2 & -6 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix}$

$|A| = 0$, A^{-1} does not exist.

$\therefore$ the system of equations does not have a unique solution.

Definition 21 (Minor) *If we choose any r rows and r columns from any matrix A by deleting all others rows and columns, then the determinants of these matrix $(r \times r)$ elements is called the minor of order r.*

Definition 22 (Rank of a matrix) *A matrix is said to be of rank r when,*

(i) It has at least one non-zero minor of order r

(ii) Every minor of order higher than r vanishes.

Examples

1. If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then $r(A) = 0$ (All the determinants are zero)

2. If $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ then $r(A) = 2$ ($|A| = 6 - 4 = 2 \neq 0$)

3. If $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ then $r(A) = 1$ ($|A| = 4 - 4 = 0$)

4. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix}$ then $r(A) = 2$ ($\because$ $|A| = 0$ and $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3 \neq 0$)

5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 8 \\ 3 & 2 & 1 \end{bmatrix}$ then $r(A) = 3$ ($\because$ $|A| \neq 0$)

6. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ then $r(A) = 1$ ($\because$ all the three sub-matrices of order 2 have determinants zero i.e. $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$)

7. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ then $r(A) = 2$ ($\because$ $|A| = 0$ and $\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} = 8 - 2 = 6 \neq 0$)

8. If $A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$ then $r(A) = 1$ ($\because$ $|A| = 0$ and determinants of all minors of 2 are also zero.)

1.2 MCQ with Answers

(1) Define: Orthogonal Matrix

Ans: If A is square matrix and $A^T A = AA^T = I$ then A is known as orthogonal matrix.

(2) what is the rank of the identity matrix of order 3?

(A) 0 (B) 1 (C) 2 (D) 3

Ans: 3

(3) Identify the type of matrix if $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$

Ans: Skew-symmetric matrix.

(4) For any matrix A the matrix $A + A^T$ is a symmetric matrix (True or False)

Ans: True.

(5) For any matrix A , $AA^{-1} = I$ (True or False)

Ans: True.

(6) Give an example of matrix A such that $A^T = -A$

Ans: $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.

(7) What is x if $A = \begin{bmatrix} 1 & 4 \\ 2 & x \end{bmatrix}$ is a singular matrix?

Ans: $x = 8$

(8) The rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(A) 0 (B) 1 (C) 2 (D) 3

Ans: 1

(9) For any matrix A and B , $(AB)^T = A^T B^T$ (True or False)

Ans: True.

(10) What is x if $A = \begin{bmatrix} 3 & 2x \\ 1-3x & 2 \end{bmatrix}$ is a symmetric matrix?

Ans: $x = \frac{1}{5}$

(11) For 2×2 matrix if the determinant is zero, then A^{-1} does not exists (True or False)

Ans: True.

(12) What is the necessary condition for multiplying two matrix?

Ans: If number of columns of columns of matrix A must be equal to the number of rows of matrix B then we can find AB.

(13) If A and B matrices with same order, then $(A+B)^T = \ldots.$

(A) $A^T + B^T$ (B) AB (C) $A+B$ (D) $A^T + B$

Ans: $A^T + B^T$

(14) What is the order of AB, if A 3 rows and 5 columns and B has 5 rows and 4 columns.

Ans: 3×4.