Fundamental Mathematics Concepts

CC-104

Unit-1: Set theory and Functions

Contents

1 Set theory

1.1 Basic of set theory

Definition 1 (Set) *A set is a well defined collection of objects.*

Usually denoted by capital letters A, B, C, X, Y, Z, etc.

Basic examples of sets

- The set of members in your family.
- The set of states in India

There are two methods of representing a set :

1. Roster or tabular form or listing method.

2. Set-builder form or property method.

For example: Write the solution set of the equation $x^2 + x - 2 = 0$.

Roster form: $\{1,\ 2\}$.

Set builder form: $\{x$: x is roots of $x^2 + x - 2 = 0\}$

We give below a few more examples of sets used particularly in mathematics, viz.

$\mathbb{N}$: the set of all natural numbers

$\mathbb{Z}$: the set of all integers

$\mathbb{Q}$: the set of all rational numbers

$\mathbb{R}$: the set of real numbers

$\mathbb{Z}^+$: the set of positive integers

$\mathbb{Q}^+$: the set of positive rational numbers, and

$\mathbb{R}^+$: the set of positive real numbers.

Definition 2 (Empty set or Null set) *A set which does not contain any element is called the empty set or the null set or the void set.*

For example:

Let A $= \{x : 1 < x < 2,\ x$ is a natural number$\}$. Then A is the empty set, because there is no natural number between 1 and 2.

Definition 3 (Finite and Infinite Sets) *A set which is empty or consists of a definite number of elements is called finite otherwise, the set is called infinite.*

For example:

Let W be the set of the days of the week. Then W is finite.

Let G be the set of points on a line. Then G is infinite.

Note: All infinite sets cannot be described in the roster form. For example, the set of real numbers cannot be described in this form, because the elements of this set do not follow any particular pattern.

Definition 4 (Subset) *A set A is said to be a subset of a set B if every element of A is also an element of B.*
The symbol $\subseteq$ stands for "is a subset or equal to of" or "is contained in".
$\rightarrow A \subseteq B$ *and* $B \subseteq A \Leftrightarrow A = B$.

Properties

- Every set is a subset of it self.
- If $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$
- The null set is a subset of every set. i.e. For any set A, $\phi \subseteq A$.

  Definition 5 (Proper Subset of a set) *Any subset A of the set B is called the proper sunset of set B, if their is at least one element of set B which does not belong to set A.*
  It is denoted by $A \subset B$.

  e.g. If $A = \{1, 2\}$, $B = \{1, 2, 3\}$ then set A is subset of set B, and its proper.

Definition 6 (Singleton set) *A set with its single member then it is called singleton set.*

e.g. $A = \{x/x \text{ is an integer between 4 and 6}\}$

Definition 7 (Equal sets) *Two sets A and B are said to be equal if they have exactly the same elements and we write $A = B$. Otherwise, the sets are said to be unequal and we write $A \neq B$.*

For example:

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 1, 4, 2\}$. Then $A = B$.

Definition 8 (Equivalent sets) *Two sets A and B are sets with the same number of elements then they are called equivalent sets. It is denoted by $A \equiv B$*

For example:

If $A = \{1, 2, 3\}$ and $B = \{4, 5, 8\}$ then we say that A and B are equivalent.

Note: Equal sets are always equivalent but converse is not true.

Definition 9 (Power Set) *The collection of all subsets of a set A is called the power set of A. It is denoted by $P(A)$. In $P(A)$, every element is a set.*
In general, if A is a set with $n(A) = m$, then it can be shown that $n[P(A)] = 2^m$.

For example: if $A = \{1, 2\}$, then $P(A) = \{f, \{1\}, \{2\}, \{1, 2\}\}$ Also, note that $n[P(A)] = 4 = 2^2$

Definition 10 (Universal Set) *A main set from which all the different subsets are considered is called universal set. It is usually denoted by U.*

e.g. If A=set of boy students. If B=set of girl students
then U=set of all students in the class.

Definition 11 (Disjoint Sets) *No elements are common in two sets then they are called disjoint.*

e.g. $A = \{a, b, c\}$ and $B = \{d, e, f\}$
Here A and B are disjoint.

1.2 Set operations

Definition 12 (Union of two sets) *The union of two sets A and B is the set $\cup$ which consists of all those elements which are either in A or in B (including those which are in both). In symbols, we write. $A \cup B = \{x : x \in A \, or \, x \in B\}$.*

The union of two sets can be represented by a Venn diagram as shown in Figure 1.
The shaded portion in Figure 1 represents $A \cup B$.

For example:

Let $A = \{2, 4, 6, 8\}$ and $B = \{6, 8, 10, 12\}$ then $A \cup B = \{2, 4, 6, 8, 10, 12\}$

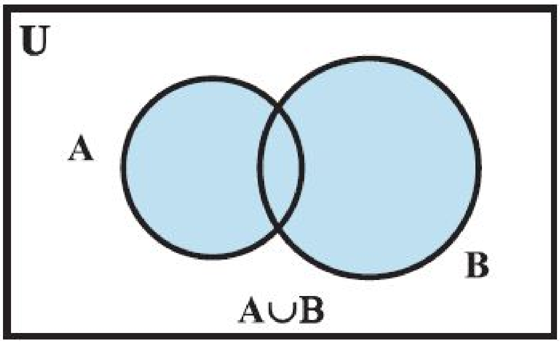


Figure 1: Venn diagram of $A \cup B$

Properties of two sets

- $A \cup A = A$ (Idempotent property)
- $A \cup \phi = A$
- $A \cup U = U$
- $A \subset (A \cup B)$; $B \subset (A \cup B)$
- $A \cup B = B \cup A$ (Commutative property)
- $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative property)

Definition 13 (Intersection of two sets) *The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write* $A \cap B = \{x : x \in A \, and \, x \in B\}$

The shaded portion in Figure 2 indicates the intersection of A and B.

Properties of intersection of two sets

- $A \cap A = A$ (Idempotent property)
- $A \cap \phi = \phi$
- $A \cap U = A$
- $(A \cap B) \subset A$; $(A \cap B) \subset B$
- $A \cap B = B \cap A$ (Commutative property)
- $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative property)

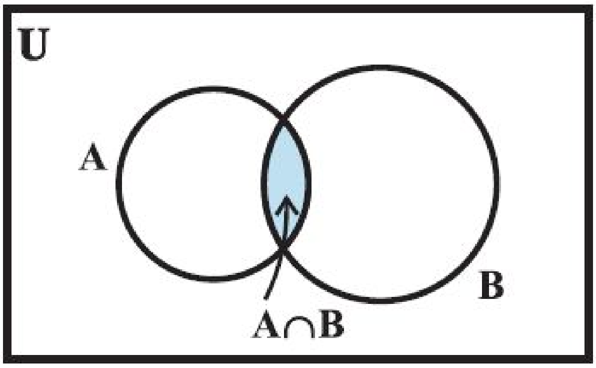


Figure 2: Intersection of two sets A and B

Definition 14 (Complement of a Set) *Let U be the universal set and A a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. Symbolically, we write A' to denote the complement of A with respect to U. Thus, $A' = \{x : x \in U \, but \, x \notin A\}$. Obviously $A' = U - A$.*

The shaded portion in Figure 3 indicates the compliment of A.

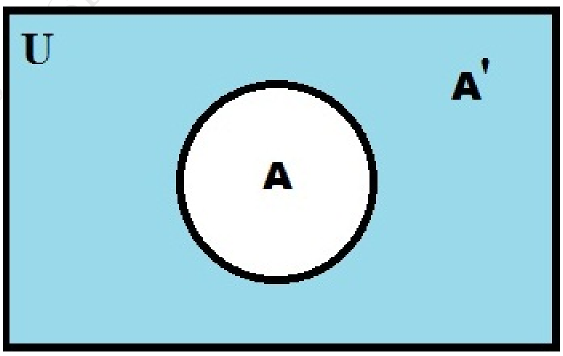


Figure 3: Complement (A') of a Set A

Properties of compliment set

- $(A')' = A$ (Idempotent property)
- $A \cap A' = \phi$

- $A \cup A' = U$
- $\phi' = U$
- $U' = \phi$
- $(A \cup B) \cap (A \cup B') = A$
- $(A \cap B) \cup (A \cap B') = A$

Definition 15 (Symmetric Difference of sets) *The symmetric difference of the sets A and B in this order is the set of elements which belong to A or B but not in both. Symbolically, we write $A \Delta B$.*

$$\begin{aligned} \therefore A - B &= \{x/x \in (A \cup B) \,\text{but}\, x \notin (A \cap B)\} \\ &= (A \cup B) - (A \cap B) \\ &= (A - B) \cup (B - A) \end{aligned}$$

The shaded portion in Figure 4 indicates the symmetric difference of A and B.

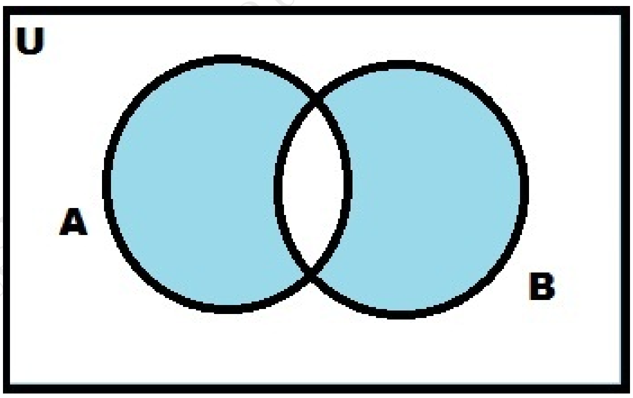


Figure 4: Symmetric difference of A and B

For example:

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8\}$ then $A - B = \{1, 3, 5\}$ and $B - A = \{8\}$.

De-Morgan's law

De-Morgan's law for union: $(A \cup B)' = A' \cap B'$

De-Morgan's law for intersection: $(A \cap B)' = A' \cup B'$

Definition 16 (Cartesian Products of Sets) *Given two non-empty sets P and Q. The cartesian product is the set of all ordered pairs of elements from P and Q, i.e., $P \times Q = \{(p, q) : p \in P, q \in Q\}$*
If either P or Q is the null set, then $P \times Q$ will also be empty set, i.e., $P \times Q = \phi$.

For example:
A is a set of 2 colours and B is a set of 3 objects, i.e., $A = \{\text{red}, \text{blue}\}$ and $B = \{b, c, s\}$, where b, c and s represent a particular bag, coat and shirt, respectively.
How many pairs of coloured objects can be made from these two sets?
Proceeding in a very orderly manner, we can see that there will be 6 distinct pairs as given below:
(red, b), (red, c), (red, s), (blue, b), (blue, c), (blue, s).

Example 1 *Write down the elements of the following sets:*

1. $A = \{x/x \text{ is prime number} < 10,\ x \in \mathbb{N}\}$
2. $B = \{x/x^6 = 1,\ x \in \mathbb{Z}\}$
3. $C = \{x/x^2 - 3x + 2 = 0,\ x \in \mathbb{R}\}$
4. $D = \{x/x \text{ is an odd number between 13 and 15}\}$
5. $E = \{2x/ - 2 \leq x < 2,\ x \in \mathbb{Z}\}$

Solution:
(1) $A = \{2, 3, 5, 7\}$
(2) $B = \{-1., 1\}$
(3) $C = \{1, 2\}$
(4) $D = \{\}$
(5) $E = \{-4, -2, 0, 2\}$

Example 2 *State the following statements are true or false, for the set $A = \{\{1, 2, 3\}, \{7, 8\}, \{5, 6, 9\}\}$*

1. $7 \in A$
2. $\{5, 6, 8\} \subseteq A$
3. $\{1, 2, 3\} \in A$

4. $\{\{7,8\}\} \subseteq A$

5. $\phi \in A$

Solution:

(1) $7 \in A$, False ($\because$ 7 is not a element of set A.)

(2) $\{5,6,9\} \subseteq A$, False ($\because$ {5,6,8} is not a element of set A.)

(3) $\{1,3,3\} \in A$, True ($\because$ $\{1,3,3\}$ is a element of set A.)

(4) $\{\{7,8\}\} \subseteq A$, True ($\because$ $\{\{7,8\}\}$ is a element of set A and each element is a subset of set.)

(5) $\phi \in A$, False, ($\because$ ϕ is not a element of set A.)

Example 3 *State the following statements are true or false:*

1. $\{3\} \subset \{1,2,\{3\}\}$

2. $\phi \neq \{\phi\}$

3. $If A = \{1,2,3\}, B = \{1,2,3,4\}$ *then* $A - B = \phi$

4. $If A = \{-1,0,1\}$ *then* $\{1\} \in A$

5. $\phi \cap U = U$

Solution:

(1) False, ($\because$ $\{3\}$ is not a element of set A, so can write $\{3\} \in \{1,2,\{3\}\}$. It is not a subset of a given set.)

(2) True

(3) True

(4) False, $\{1\}$ is a subset of set A while we can write $1 \in A$

(5) False. $\phi \cap U = \phi$

Example 4 *If* $A = \{1,2,3\}$, $B = \{-1,0,1,2\}$, $C = \{2,3,4,5\}$. *Find* $A \cup B$, $B \cup C$, $C \cup A$, $A \cup B \cup C$, $A \cap B$, $B \cap C$, $C \cap A$, $A \cap B \cap C$, $A \cup (B \cap C)$, $A \cap (B \cup C)$, $(A \cup B) \cap C$, $A - B$, $B - C$, $A \Delta B$, $(A \Delta B)\Delta C$

Solution:

$$
\begin{aligned}
A \cup B &= \{1, 2, 3\} \cup \{-1, 0, 1, 2\} \\
&= \{-1, 0, 1, 2, 3\}
\end{aligned}
$$

$$
\begin{aligned}
B \cup C &= \{-1,0,1,2\} \cup \{2,3,4,5\} \\
&= \{-1,0,1,2,3,4,5\} \\
C \cup A &= \{2,3,,4,5\} \cup \{1,2,3\} \\
&= \{1,2,3,4,5\} \\
A \cup B \cup C &= \{1,2,3\} \cup \{-1,0,1,2\} \cup \{2,3,4,5\} \\
&= \{-1,0,1,2,3,4,5\} \\
A \cap B &= \{1,2,3\} \cap \{-1,0,1,2\} \\
&= \{1,2\} \\
B \cap C &= \{-1,0,1,2\} \cap \{2,3,4,5\} \\
&= \{2\} \\
C \cap A &= \{2,3,,4,5\} \cap \{1,2,3\} \\
&= \{2,3\} \\
A \cap B \cap C &= \{1,2,3\} \cap \{-1,0,1,2\} \cap \{2,3,4,5\} \\
&= \{2\} \\
A \cup (B \cap C) &= \{1,2,3\} \cup (\{-1,0,1,2\} \cap \{2,3,4,5\}) \\
&= \{1,2,3\} \cup \{2\} \\
&= \{1,2,3\} \\
A \cap (B \cup C) &= \{1,2,3\} \cap (\{-1,0,1,2\} \cup \{2,3,4,5\}) \\
&= \{1,2,3\} \cap \{-1,0,1,2,3,4,5\} \\
&= \{1,2,3\} \\
(A \cup B) \cap C &= (\{1,2,3\} \cup \{-1,0,1,2\}) \cap \{2,3,4,5\} \\
&= \{-1,0,1,2,3\} \cap \{2,3,4,5\} \\
&= \{2,3\} \\
A - B &= \{1,2,3\} - \{-1,0,1,2\} \\
&= \{3\} \\
B - C &= \{-1,0,1,2\} - \{2,3,4,5\} \\
&= \{-1,0,1\}
\end{aligned}
$$

$$
\begin{aligned}
A\Delta B &= (A\cup B) - (A\cap B)\\
&= \{-1,0,1,2,3\} - \{1,2\}\\
&= \{-1,0,3\}\\
(A\Delta B)\Delta C &= [(A\Delta B)\cup C] - [A\Delta B)\cap C]\\
&= \{-1,0,2,3,4,5\} - \{3\}\\
&= \{-1,0,2,4,5\}
\end{aligned}
$$

Example 5 *If* $A = \{x/-3 \le x < 2,\, x \in \mathbb{Z}\}$, $B = \{y/y \le 5,\, y \in \mathbb{N}\}$, $C = \{z/z^2 < 10,\, z \in \mathbb{N}\}$. *Then verify the following results.*

1. $A\cup(B\cup C) = (A\cup B)\cup C$
2. $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$
3. $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$
4. $A-(B\cap C) = (A-B)\cup(A-C)$
5. $A-(B\cup C) = (A-B)\cap(A-C)$

Solution:

Here, $A = \{-3,-2,-1,0,1\}$, $B = \{1,2,3,4,5\}$, $C = \{1,2,3\}$.

(1) $A\cup(B\cup C) = (A\cup B)\cup C$

$$
\begin{aligned}
L.H.S &= A\cup(B\cup C)\\
&= \{-3,-2,-1,0,1\}\cup\{1,2,3,4,5\}\\
&= \{-3,-2,-1,0,1,2,3,4,5\} \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
R.H.S &= (A\cup B)\cup C\\
&= \{-3,-2,-1,0,1,2,3,4,5\}\cup\{1,2,3\}\\
&= \{-3,-2,-1,0,1,2,3,4,5\} \qquad (2)
\end{aligned}
$$

From (1) and (2) we get, L.H.S=R.H.S.

Hence, $A \cup (B \cup C) = (A \cup B) \cup C$

(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$\begin{aligned} L.H.S &= A \cap (B \cup C) \\ &= \{-3,-2,-1,0,1\} \cap \{1,2,3,4,5\} \\ &= \{1\} \qquad (3) \end{aligned}$$

$$\begin{aligned} R.H.S &= (A \cap B) \cup (A \cap C) \\ &= \{1\} \cup \{1\} \\ &= \{1\} \qquad (4) \end{aligned}$$

From (3) and (4) we get, L.H.S=R.H.S.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(3) $A \cap (B \cap C) = (A \cap B) \cap C$

$$\begin{aligned} L.H.S &= A \cap (B \cap C) \\ &= \{-3,-2,-1,0,1\} \cap \{1,2,3\} \\ &= \{1\} \qquad (5) \end{aligned}$$

$$\begin{aligned} R.H.S &= (A \cap B) \cap C) \\ &= \{1\} \cap \{1,2,3\} \\ &= \{1\} \qquad (6) \end{aligned}$$

From (5) and (6) we get, L.H.S=R.H.S.

Hence, $A \cap (B \cap C) = (A \cap B) \cap C$

(4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$\begin{aligned} L.H.S &= A \cup (B \cap C) \\ &= \{-3,-2,-1,0,1\} \cup \{1,2,3\} \\ &= \{-3,-2,-1,0,1,2,3\} \qquad (7) \end{aligned}$$

$$\begin{aligned} R.H.S &= (A \cup B) \cap (A \cup C) \\ &= \{\{-3,-2,-1,0,1,2,3,4,5\}\} \cap \{-3,-2,-1,0,1,2,3\} \\ &= \{-3,-2,-1,0,1,2,3\} \qquad (8) \end{aligned}$$

From (7) and (8) we get, L.H.S=R.H.S.

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(5) $A - (B \cap C) = (A - B) \cup (A - C)$

$$\begin{aligned} L.H.S &= A - (B \cap C) \\ &= \{-3, -2, -1, 0, 1\} - \{1, 2, 3\} \\ &= \{-3, -2, -1, 0\} \qquad (9) \end{aligned}$$

$$\begin{aligned} R.H.S &= (A - B) \cup (A - C) \\ &= \{-3, -2, -1, 0\} \cup \{-3, -2, -1, 0\} \\ &= \{-3, -2, -1, 0\} \qquad (10) \end{aligned}$$

From (9) and (10) we get, L.H.S=R.H.S.

Hence, $A - (B \cap C) = (A - B) \cup (A - C)$

(6) $A - (B \cup C) = (A - B) \cap (A - C)$

$$\begin{aligned} L.H.S &= A - (B \cup C) \\ &= \{-3, -2, -1, 0, 1\} - \{1, 2, 3, 4, 5\} \\ &= \{-3, -2, -1, 0\} \qquad (11) \end{aligned}$$

$$\begin{aligned} R.H.S &= (A - B) \cap (A - C) \\ &= \{-3, -2, -1, 0\} \cap \{-3, -2, -1, 0\} \\ &= \{-3, -2, -1, 0\} \qquad (12) \end{aligned}$$

From (11) and (12) we get, L.H.S=R.H.S.

Hence, $A - (B \cup C) = (A - B) \cap (A - C)$

Example 6 *Let* $A = \{a, b\}$, $B = \{p, q\}$ *then find* $A \times B$, $B \times A$, $A \times A$ *and* $B \times B$.

Solution:

$$\begin{aligned} A \times B &= \{a, b\} \times \{p, q\} \\ &= \{(a, p), (a, q), (b, p), (b, q)\} \\ B \times A &= \{p, q\} \times \{a, b\} \\ &= \{(p, a), (p, b), (q, a), (q, b)\} \end{aligned}$$

$$
\begin{aligned}
A \times A &= \{a, b\} \times \{a, b\} \\
&= \{(a, a), (a, b), (b, a), (b, b)\} \\
B \times B &= \{p, q\} \times \{p, q\} \\
&= \{(p, p), (p, q), (q, p), (q, q)\}
\end{aligned}
$$

Example 7 *Let* $A = \{1, 2, 3\}$, $B = \{2, 4\}$ *and* $C = \{1, 3, 5\}$, *find* $(A \cup B) \times C$, $A \times (B \cup C)$, $B \times C$ *and* $(A \cup B) \times (A \cap C)$.

Solution:

$$
\begin{aligned}
(A \cap B) \times C &= \{2\} \times \{1, 3, 5\} \\
&= \{(2, 1), (2, 3), (2, 5)\} \\
B \times (B \cup C) &= \{1, 2, 3\} \times \{1, 2, 3, 4, 5\} \\
&= \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), \\
&(2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5)\} \\
B \times C &= \{2, 4\} \times \{1, 3, 5\} \\
&= \{(2, 1), (2, 3), (2, 5), (4, 1), (4, 3), (4, 5)\} \\
(A \cup B) \times (A \cap C) &= \{1, 2, 3, 4\} \times \{1, 3\} \\
&= \{(1, 1), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}
\end{aligned}
$$

Example 8 *If*

U = *Set of letters of the word 'WHAT'*

A = *Set of letters of the word 'HET'*

B = *Set of letters of the word 'HEAR'*

C = *Set of letters of the word 'HOW'*

Find $(A \cap B) \times (B \cap C)$, $A \cap (B - C)$, $(A - B)' \cap C'$ *and* $(A \cap B \cap C)'$

Solution:

Here, $U = \{W, H, A, T\}$, $A = \{H, T\}$, $B = \{H, A\}$, $C = \{H, W\}$.

$$\begin{aligned}
(A \cap B) \times (B \cap C) &= \{H\} \times \{H\} \\
&= \{H, H\} \\
A \cap (B - C) &= \{H, T\} \cap \{A\} \\
&= \phi \\
(A - B)' \cap C' &= \{T\}' \cap \{H, W\}' \\
&= \{W, H, A\} \cap \{A, T\} \\
&= \{A\} \\
(A \cap B \cap C)' &= \{H\}' \\
&= \{W, A, T\}
\end{aligned}$$

1.3 Cardinality or size of a set

If A is a non-empty set. Then the number of elements of set A is denoted by $n(A)$.
e.g. If $A = \{1, 5, 9\}$ then $n(A) = 3$.
Important results

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ (if A and B are not disjoint)
- $n(A \cup B) = n(A) + n(B)$ (if A and B are disjoint)
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C)$ (if A and B are disjoint)
- $n(A) = n(A - B) + n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$
- $n(A \cup B)' = n(U) - n(A \cup B)$

Example 9 *The result of students of two classes of a school with 120 students in the subject of Mathematics and Physics are obtained as follows:*

- *The no. of students passing in Mathematics is 55.*
- *The no. of students passing in Physics is 60.*

- *The no. of students passing in both the subjects is 22.*

Answer the following questions.

1. *How many students are passing only in Mathematics?*
2. *How many students are passing only in Physics?*
3. *How many students are fail in both the subjects?*

Solution:

Let M be the no. of passing students in Mathematics and P be the no. of passing students in Physics.

$\therefore\ n(M) = 55$, $n(P) = 60$ and $n(M \cap P) = 22$.

$$n(M \cup P) = n(M) + n(P) - n(M \cap P)$$
$$n(M \cup P) = 55 + 60 - 22$$
$$n(M \cup P) = 93$$

Hence, total 93 students passes either in Mathematics or in Physics or in both.

So, no. of students fail in both the subject is,

$$\begin{aligned} n(M \cup P)' &= n(U) - n(M \cup P) \\ &= 120 - 93 \\ &= 27 \end{aligned}$$

Also, no. of students passing only in Mathematics is $n(M - P)$

$$n(M) = n(M - P) + n(M \cap P)$$
$$\therefore 55 = n(M - P) + 22$$
$$\therefore n(M - P) = 33$$

No. of students passing only in Physics is $n(P - M)$

$$n(P) = n(P - M) + n(M \cap P)$$
$$\therefore 60 = n(P - M) + 22$$
$$\therefore n(P - M) = 38$$

Example 10 *If* $A = \{1,2,3,4,5,6\}$, $B = \{2,5,6,8\}$ *and* $C = \{1,6,7\}$ *then prove that* $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

Solution:

From the given sets it is clear that $n(A) = 6$, $n(B) = 4$ and $n(C) = 3$

from the given sets $(A \cup B \cup C) = \{1,2,3,4,5,6,7,8\}$

$A \cap B = \{2,5,6\}$, $B \cap C = \{6\}$, $A \cap C = \{1,6\}$, and $A \cap B \cap C = \{6\}$

$n(A \cap B) = 3$, $n(B \cap C) = 1$, $n(A \cap C) = 2$ and $n(A \cap B \cap C) = 1$

Therefore,

$$n(A \cup B \cup C) = 8 \tag{13}$$

$$\begin{aligned} & n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C) \\ & = 6 + 4 + 3 - 3 - 1 - 2 + 1 \\ & = 8 \end{aligned} \tag{14}$$

From (13) and (14) we get,

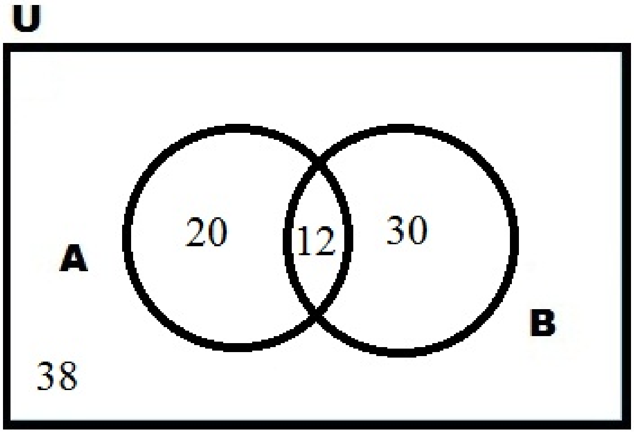
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

Example 11 *IF* $n(U) = 100$, $n(A) = 32$, $n(B) = 42$, $n(A \cup B) = 62$, *then find* $n(A' \cup B')$ *and* $n(A \cup B')$.

Solution:

$n(A) = 32$, $n(B) = 42$, $n(A \cup B) = 62$

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 62 &= 32 + 42 - n(A \cap B) \\ n(A \cap B) &= 74 - 62 \\ n(A \cap B) &= 12 \end{aligned}$$



(i)

$$\begin{aligned} n(A' \cup B') &= n(A \cap B)' \\ &= n(U) - n(A \cap B) \\ &= 100 - 12 \\ &= 88 \end{aligned}$$

(ii)

$$\begin{aligned} n(A \cup B') &= n(A - B) + n(A \cup B)' \\ &= 20 + 38 \\ &= 58 \end{aligned}$$

Example 12 *If* $U = \{-3, -1, 0, 1, 3\}$, $A = \{-3, -1, 1\}$, $B = \{-1, 1, 3\}$ *and* $C = \{-1, 0, 1\}$ *then verify the following*

1. $(A \cup B)' = A' \cap B'$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
3. $B - A = A' \cap B = B - (A \cap B)$
4. $A \Delta B = (A - B) \cup (B - A)$
5. $(B')' = B$
6. $A \cup B = A \cup [B - (A \cap B)]$.

Solution:

(1) $(A \cup B)' = A' \cap B'$

$$\begin{aligned} U &= \{-3, -1, 0, 1, 3\} \\ A &= \{-3, -1, 1\} \\ B &= \{-1, 1, 3\} \\ A \cup B &= \{-3, -1, 1, 3\} \\ (A \cup B)' &= \{0\} \end{aligned} \tag{15}$$

and

$$\begin{aligned} A' = \{0, 3\},\ B' &= \{-3, 0\} \\ A' \cap B' &= \{0\} \end{aligned} \tag{16}$$

From (15) and (16) we get, $(A \cup B)' = A' \cap B'$

(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Here $B \cap C = \{-1, 1\}$

$$A \cup (B \cap C) = \{-3, -1, 1\} \tag{17}$$

and

$$\begin{aligned} A \cup B &= \{-3, -1, 1, 3\} \\ A \cup C &= \{-3, -1, 0, 1\} \\ (A \cup B) \cap (A \cup C) &= \{-3, -1, 1\} \end{aligned} \tag{18}$$

From (17) and (18) we get, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(3) $B - A = A' \cap B = B - (A \cap B)$

$$B - A = \{3\} \tag{19}$$

$$\begin{aligned} A' &= \{0, 3\} \\ A' \cap B &= \{3\} \end{aligned} \tag{20}$$

$$\begin{aligned} A \cap B &= \{-1, 1\} \\ B - (A \cap B) &= \{3\} \end{aligned} \tag{21}$$

From (19), (20) and (21) we get, $B - A = A' \cap B = B - (A \cap B)$

(4) $A\Delta B = (A - B) \cup (B - A)$

$$\begin{aligned} A \cup B &= \{-3, -1, 1, 3\} \\ A \cap B &= \{-1, 1\} \\ A\Delta B &= (A \cup B) - (A \cap B) = \{-3, 3\} \end{aligned} \tag{22}$$

$$\begin{aligned} A - B &= \{-3\} \\ B - A &= \{3\} \\ (A - B) \cup (B - A) &= \{-3, 3\} \end{aligned} \tag{23}$$

From (22) and (23) we get, $A\Delta B = (A - B) \cup (B - A)$

(5) $(B')' = B$

$$B = \{-1, 1, 3\} \tag{24}$$

$$\begin{aligned} \therefore B' &= \{-3, 0\} \\ \therefore (B')' &= \{-1, 1, 3\} \end{aligned} \tag{25}$$

From (24) and (25) we get, $(B')' = B$

(6) $A \cup B = A \cup [B - (A \cap B)]$

$$A \cup B = \{-3, -1, 1, 3\} \tag{26}$$

$$\begin{aligned} A \cap B &= \{-1, 1\} \\ B - (A \cap B) &= \{3\} \\ A \cup [B - (A \cap B)] &= \{-3, -1, 1, 3\} \end{aligned} \tag{27}$$

From (26) and (27) we get, $A \cup B = A \cup [B - (A \cap B)]$

Example 13 *If* $A = \{5, 6, 7\}$, $B = \{7, 8\}$ *and* $C = \{5, 8\}$ *then verify*

1. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

2. $A \times (B - C) = (A \times B) - (A \times C)$

3. $A \times (B \Delta C) = (A \times B) \Delta (A \times C)$

Solution:

(1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Here, $B \cap C = \{8\}$

$$A \times (B \cap C) = \{(5,8),(6,8),(7,8)\} \tag{28}$$

$$\begin{aligned} A \times B &= \{(5,7),(5,8),(6,7),(6,8),(7,7),(7,8)\} \\ A \times C &= \{(5,5),(5,8),(6,5),(6,8),(7,5),(7,8)\} \\ (A \times B) \cap (A \times C) &= \{(5,8),(6,8),(7,8)\} \end{aligned} \tag{29}$$

From (28) and (29) we get, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(2) $A \times (B - C) = (A \times B) - (A \times C)$

Here, $B - C = \{7\}$

$$A \times (B - C) = \{(5,7),(6,7),(7,7)\} \tag{30}$$

$$\begin{aligned} A \times B &= \{(5,7),(5,8),(6,7),(6,8),(7,7),(7,8)\} \\ A \times C &= \{(5,5),(5,8),(6,5),(6,8),(7,5),(7,8)\} \\ (A \times B) - (A \times C) &= \{(5,7),(6,7),(7,7)\} \end{aligned} \tag{31}$$

From (30) and (31) we get, $A \times (B - C) = (A \times B) - (A \times C)$

(3) $A \times (B \Delta C) = (A \times B) \Delta (A \times C)$

Here, $B \cup C = \{5,7,8\}$ and $B \cap C = \{8\}$

$$\begin{aligned} B \Delta C &= (B \cup C) - (B \cap C) = \{5,7\} \\ A \times (B \Delta C) &= \{(5,5),(5,7),(6,5),(6,7),(7,5),(7,7)\} \end{aligned} \tag{32}$$

$$
\begin{aligned}
A \times B &= \{(5,7),(5,8),(6,7),(6,8),(7,7),(7,8)\} \\
A \times C &= \{(5,5),(5,8),(6,5),(6,8),(7,5),(7,8)\} \\
(A \times B) \cup (A \times C) &= \{(5,5),(5,7),(5,8),(6,5),(6,7),(7,5),(7,7)\} \\
(A \times B) \cap (A \times C) &= \{(5,8),(6,8),(7,8)\} \\
(A \times B)\Delta(A \times C) &= [(A \times B) \cup (A \times C)] - [(A \times B) \cap (A \times C)] \\
&= \{(5,5),(5,7),(6,5),(6,7),(7,5),(7,7)\} \qquad (33)
\end{aligned}
$$

From (32) and (33) we get, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

2 Functions

2.1 Introduction

In many instances we assign to each element of a set a particular element of a second set (which may be the same as the first). For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set A,B,C,D, F. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens. This assignment of grades is illustrated in Figure 5.

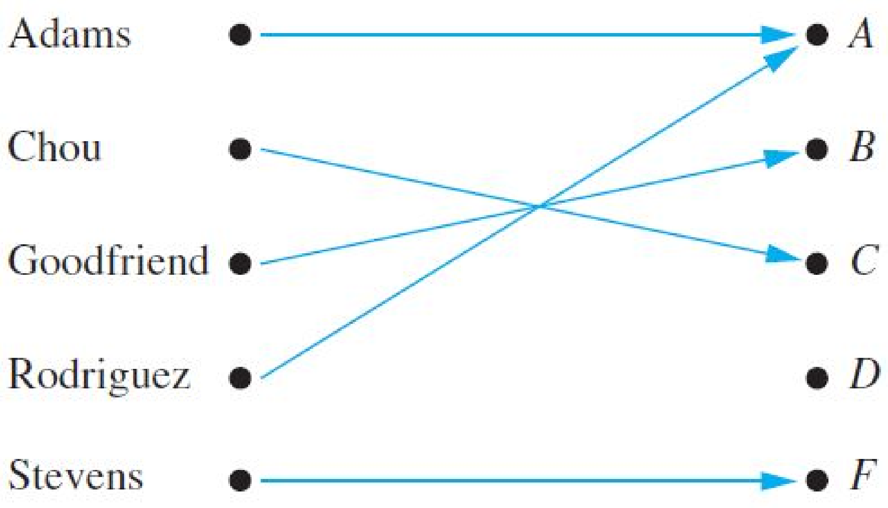


Figure 5: Assignment of Grades in a Mathematics Class.

This assignment is an example of a function. The concept of a function is extremely important in mathematics and computer science. For example, in discrete mathematics functions are used in the definition of such discrete structures as sequences and strings.

Functions are also used to represent how long it takes a computer to solve problems of a given size. Many computer programs and subroutines are designed to calculate values of functions. This section reviews the basic concepts involving functions needed in computer science.

Definition 17 (Function) *Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.*

Note: Functions are sometimes also called mappings or transformations.

Definition 18 *If f is a function from A to B, we say that A is the domain of f and B is the codomain of f. If $f(a) = b$, we say that b is the image of a and a is a preimage of b. The range, or image, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.*

For Example

If $f : A \to B$ is a function with $A = \{1, 3, 5\}$ and $B = \{1, 3, 5, 7\}$ then domain of the function f is A.

$\therefore D_f = \{1, 3, 5\}$

and co-domain of the function f is B. $\therefore$ co-domain$= \{1, 3, 5, 7\}$

Example 14 *Let $f : A \to B$ is a function $f(x) = x + 3$ where $A = \{1, 3, 5\}$ and $B = \{2, 4, 6, 8\}$. $1, 3, 5 \in A$ then find images of A*

Solution:

$f(x) = x + 3$

$f(1) = 1 + 3 = 4$

$f(3) = 3 + 3 = 6$

$f(5) = 5 + 3 = 8$

Here, 4,6 and 8 are the images of set A.

Example 15 *Let $f : A \to B$ is a function $f(x) = x - 1$, where $A = \{2, 3, 4\}$ and $B = \{0, 1, 2, 3, 4, 5\}$. then find range of f*

Solution:

$f(x) = x - 1$

$f(2) = 2 - 1 = 1$

$f(3) = 3 - 1 = 2$

$f(4) = 4 - 1 = 3$

$\therefore R_f = \{1, 2, 3\}$.

Here, $f : A \to B$ is a function. Set A is a domain, set B is a co-domain and $R_f = \{1, 2, 3\}$

Example 16 *Give the domain, co-domain and range of the function $f : A \to B$ defined as $f(x) = 2x$, where $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$.*

Solution:

Domain $= D_f = \{1, 2, 3\}$

Co-domain$= B = \{1, 2, 3, 4, 5, 6, 7\}$

Range$= R_f = \{f(1), f(2), f(3)\} = \{2, 4, 6\}$

Example 17 *Determine domain, co-domain and range of the function $f : A \to N$ defined as $f(x) = 3x + 1$, where $A = \{1, 2, 3, 4\}$.*

Solution:

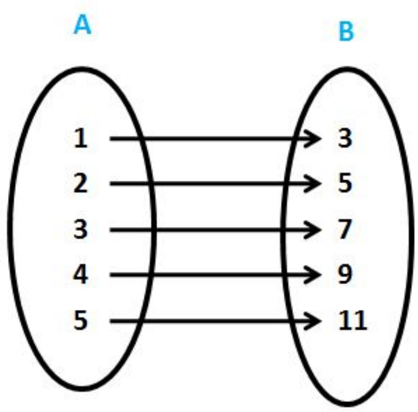
Domain $= A = \{1, 2, 3, 4\}$, Co-domain$= N$

Range$= R_f = \{f(1), f(2), f(3), f(4)\} = \{4, 7, 10, 13\}$

It is clear that R_f is a subset of co-domain N.

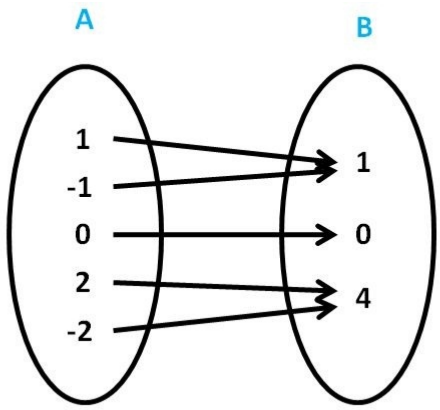
Example 18 *A few relationships are defined as bellow. Determine whether they are functions or not.*

(i)



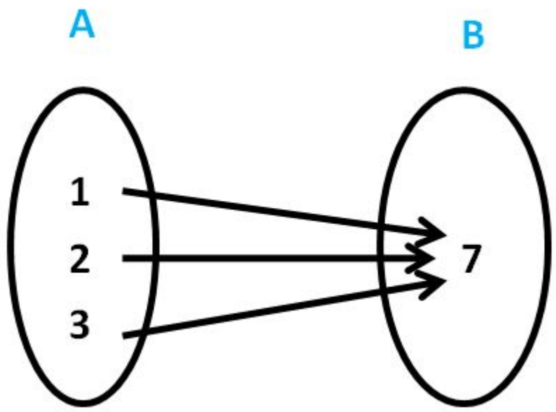
This is a function, because each element of set A is associated with element of set B.

(ii)



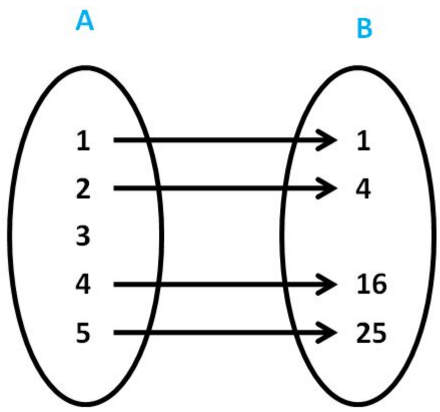
This is a function, because each element of set A is associated with element of set B.

(iii)



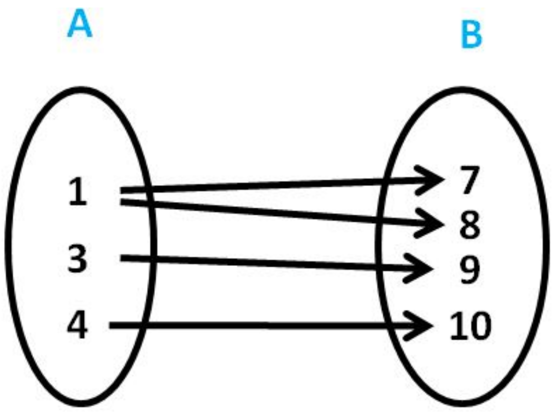
This is a function, because each element of set A is associated with one element of set B.

(iv)



This is not a function, because image of element 3 of set A is not in set B.

(v)



This is not a function, because element 1 of set A is associated with two elements 7 and 8 of set B.

Example 19 *State whether the following statements are true or false.*

1. $f : A \to B$, $A = \{1, 2, 3\}$, $B = \{3, 6, 9, 12\}$, $f(x) = 3x$ *is a function.*
2. $f : A \to B$, $A = \{-1, 0, 1, 2\}$, $B = \{1, 4, 9\}$, $f(x) = x^2$ *is a function.*
3. $f : N \to N$, $f(x) = 5x + 2$ *is a function.*
4. *The relation ship between the set of different countries of the world and their capitals is a function.*

Solution:

(1) $A = \{1, 2, 3\}$, $B = \{3, 6, 9, 12\}$, $f(x) = 3x$

Here, $f(1) = 3$, $f(2) = 6$ and $f(3) = 9$

Thus each element of set A has unique image in set B.

Hence, $f : A \to B$ is a function.

(2) $A = \{-1, 0, 1, 2\}$, $B = \{1, 4, 9\}$, $f(x) = x^2$

Here, $f(-1) = 1 \in B$, $f(0) = 0 \notin B$, $f(1) = 1 \in B$and $f(2) = 4 \in B$

Thus $0 \in A$ is not connected with any element of set B.

Hence, $f : A \to B$ is not a function.

(3) Yes, $f : N \to N$ is a function is true, because all the images are in N.

(4) The capitals of different countries of the world are definite, means each country is

uniquely associated with its capital. So, the relation ship between the set of different countries of the world and their capitals is a function.

Example 20 *If the function $f(x) = 1 - x - x^2$, then find $f(1)$, $f(-1)$ and $f(-2)$.*

Solution:

Here, $f(x) = 1 - x - x^2$

$$\begin{aligned} f(1) &= 1 - 1 - 1^2 = -1 \\ f(-1) &= 1 - (-1) - (-1)^2 = 1 + 1 - 1 = 1 \\ f(-2) &= 1 - (-2) - (-2)^2 = 1 + 2 - 4 = -1 \end{aligned}$$

Definition 19 (Equal Functions) *Let two functions f and g are defined on same domain and $f(x) = g(x), \forall x \in$ Domain then f and g are called equal functions, it is denoted by $f = g$.*

Example 21 *If the functions $f(x) = x^2$ and $g(x) = 7x - 6$ $x \in \{1, 6\}$. Is the functions f and g are equal?*

Solution:

$$\begin{aligned} f(1) = (1)^2 = 1\,; g(1) = 7(1) - 6 = 1 \\ f(6) = (6)^2 = 36\,; g(6) = 7(6) - 6 = 36 \\ \therefore f(x) = g(x)\, \forall x \in \{1, 6\} \end{aligned}$$

Hence, f and g are equal functions.

Example 22 *The function $f(x) = \frac{3x+5}{5x+3}$, then prove that $f(x) \cdot f(\frac{1}{x}) = 1$.*

Solution:

Here, $f(x) = \frac{3x+5}{5x+3}$

$$\begin{aligned} f\left(\frac{1}{x}\right) &= \frac{3\left(\frac{1}{x}\right) + 5}{5\left(\frac{1}{x}\right) + 3} = \frac{\frac{3+5x}{x}}{\frac{5+3x}{x}} = \frac{3 + 5x}{5 + 3x} \\ f(x) \cdot f\left(\frac{1}{x}\right) &= \frac{3x + 5}{5x + 3} \times \frac{3 + 5x}{5 + 3x} = 1 \end{aligned}$$

Hence proved.

Example 23 *If the functions* $f(x) = x^5 - 2x + \frac{1}{x}$*, prove that* $f(x) + f(-x) = 0$.

Solution:

$$\begin{aligned}
f(x) &= x^5 - 2x + \frac{1}{x} \\
f(-x) &= (-x)^5 - 2(-x) + \frac{1}{(-x)} \\
&= -(x)^5 + 2(x) - \frac{1}{x} \\
f(x) + f(-x) &= x^5 - 2x + \frac{1}{x} + \left[-(x)^5 + 2(x) - \frac{1}{x}\right] \\
&= x^5 - 2x + \frac{1}{x} - (x)^5 + 2(x) - \frac{1}{x} \\
&= 0
\end{aligned}$$

Hence, proved.

Definition 20 (odd function) *A function* $f : A \to B$ *is said to be odd if*
$f(-x) = -f(x), \forall x \in A$

For example

(i) The function $f(x) = x$ is odd, because $f(x) = x$ and $f(-x) = -x = -f(x)$.

(ii) The function $f(x) = 3x$ is odd, because $f(x) = 3x$ and $f(-x) = 3(-x) = -3x = -f(x)$.

Definition 21 (even function) *A function* $f : A \to B$ *is said to be even if*
$f(-x) = f(x), \forall x \in A$

For example

(i) The function $f(x) = x^4$ is even, because $f(x) = x^4$ and $f(-x) = (-x)^4 = x^4 = f(x)$.

(ii) The function $f(x) = x^2$ is even, because $f(x) = x^2$ and $f(-x) = (-x)^2 = x^2 = f(x)$.

Example 24 *Test whether the following functions are even, odd or neither.*

1. $f : R \to R$; $f(x) = -7x^2 + 2, \forall x \in R$
2. $g : R \to R$; $g(x) = 3x^3 + 4x + 12, \forall x \in R$
3. $h : R \to R$; $h(x) = 11x^5 - 7x^3 + 5x, \forall x \in R$

Solution:

(1)

$$\begin{aligned} f(x) &= -7x^2 + 2 \\ f(-x) &= -7(-x)^2 + 2 \\ &= -7x^2 + 2 \\ &= f(x) \end{aligned}$$

Hence, $f(-x) = f(x)$, so $f(x)$ is even.

(2)

$$\begin{aligned} g(x) &= 3x^3 + 4x + 12 \\ g(-x) &= 3(-x)^3 + 4(-x) + 12 \\ &= -3x^3 - 4(x) + 12 \\ &\neq g(x) \end{aligned}$$

Hence, $g(x)$ is neither odd nor even.

(3)

$$\begin{aligned} h(x) &= 11x^5 - 7x^3 + 5x \\ h(-x) &= 11(-x)^5 - 7(-x)^3 + 5(-x) \\ &= -(11x^5 - 7x^3 + 5x) \\ &= -h(x) \end{aligned}$$

Hence, $h(-x) = -h(x)$, so $h(x)$ is odd.

Definition 22 (Surjective (onto)) *A function $f : X \to Y$ is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.*
The function f_3 and f_4 in Figure 6 (iii), (iv) are onto and the function f_1 in Figure 6 (i) is not onto as elements e, f in X_2 are not the image of any element in X_1 under f_1.

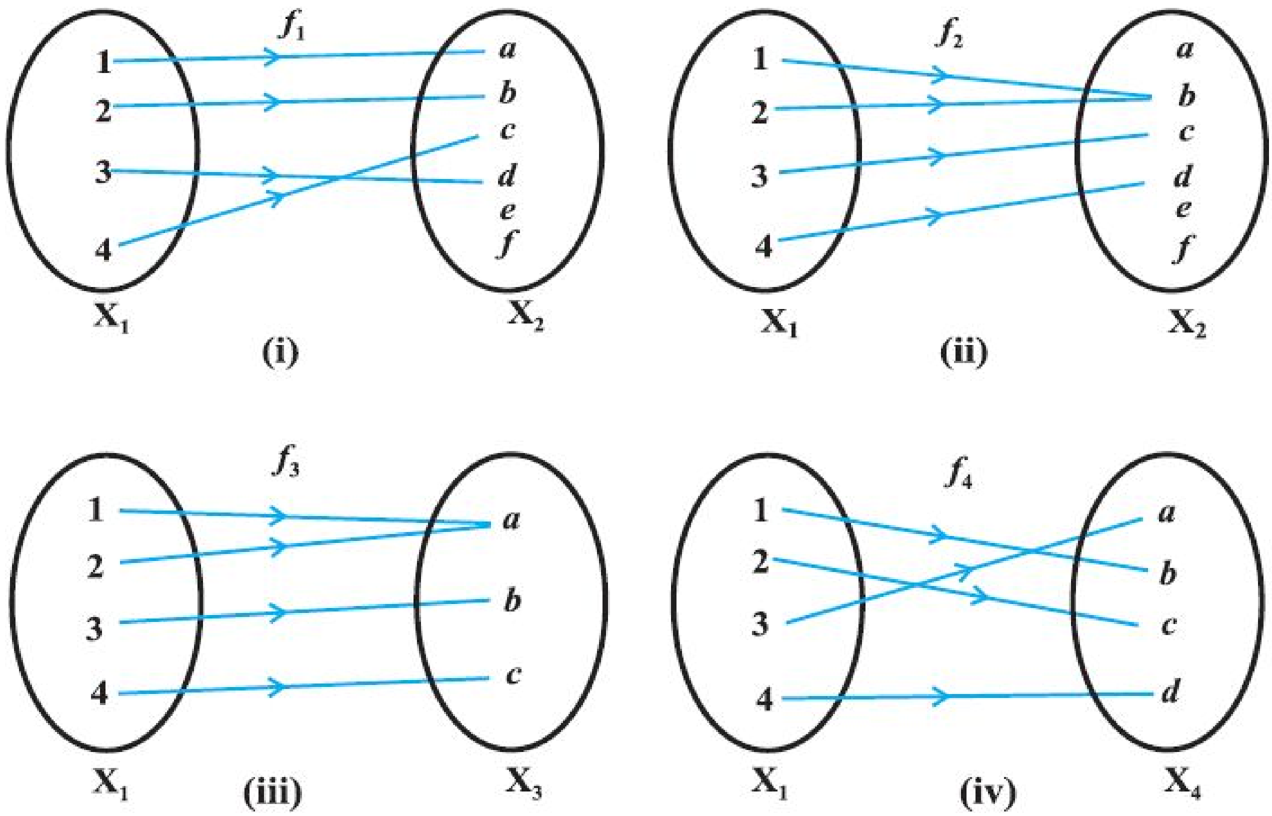


Figure 6: Types of functions

Definition 23 (Injective (one-one)) *A function $f : X \to Y$ is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e., for every $x_1\, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called many-one.*
The function f_1 and f_4 in Figure 6 (i), (iv) are one-one and the function f_2 and f_3 in Figure 11 (i), (iv) are many one.

Definition 24 (Bijective (one-one and onto)) *A function $f : X \to Y$ is said to be one-one and onto (or bijective), if f is both one-one and onto.*
The function f_4 in Figure 6 (iv) is one-one and onto.

Example 25 *Is the function $f(x) = x^2$ from the set of integers to the set of integers bijection?*

Solution

The function f is not onto and one-one because there is no integer x with $x^2 = -1$ and

$f(1) = f(-1) = 1$, but $1 \neq -1$.
Hence, f is not bijection.

Definition 25 (Into function) *A function $f : A \to B$ is an in to function, if their exists an element in B having no pre-image in A, i.e. it is not onto*

Definition 26 (Many-one function) *Suppose $f : A \to B$ is a function. The images are same in co-domain B for different elements from A then f is called many-one function.*

that means, for any $x_1, x_2 \in A$, $x_1 \neq x_2$ then $f(x_1) = f(x_2)$
e.g. Let $f : Z \to Z$ $f(x) = x^2$
so, for $-1 \neq 1$ but $f(-1) = f(-1)^2 = 1$ and $f(1) = f(1)^2 = 1$
Hence, $f(-1) = f(1)$
So, different elements of set A we have same image. Thus f is many-one function.

Definition 27 (Identity Function) *For a function $f : R \to R$, $f(x) = x, \forall x \in R$ is known as identity function.*

Definition 28 (Constant Function) *If function $f : A \to B$ defined as $f(x) = c$ on two sets A and B then $f : A \to B$ is called constant function.*

e.g. $f : N \to N$, $f(x) = 5$
$\therefore f(1) = 5 = f(2) = f(3)$......

Definition 29 (Composition of Functions) *Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g, denoted by $g \circ f$, is defined as the function $g \circ f : A \to C$ given by $g \circ f(x) = g(f(x)), \forall x \in A$.*

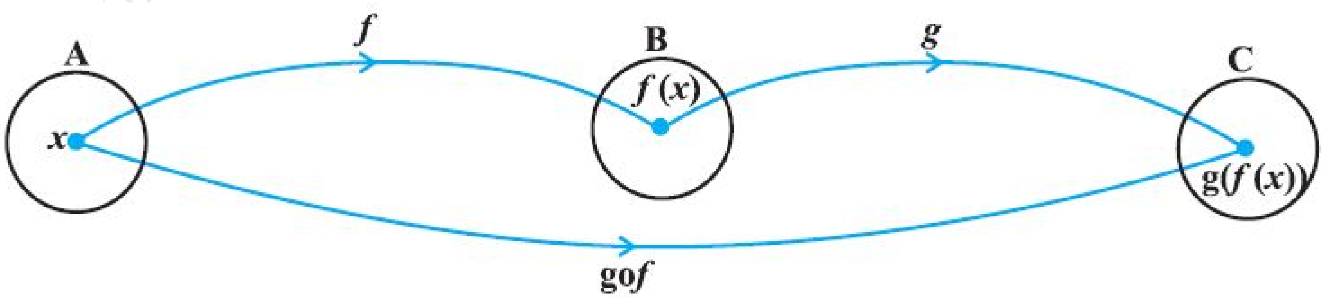


Figure 7: Composition of Functions

Definition 30 (Inverse function) *Let $f : A \to B$ is said to be inverse if and only if it is one-one and onto. it is denoted by $f^{-1} : B \to A$.*

Example 26 *If $f : R - \{0, 1\} \to R - \{0, 1\}$, $f(x) = 1 - \frac{1}{x}$, find $f \circ f \circ f(x)$.*

Solution:

$$\begin{aligned}
f \circ f(x) &= f[f(x)] \\
&= f\left(1 - \frac{1}{x}\right) \\
&= 1 - \frac{1}{1 - \frac{1}{x}} \\
&= 1 - \frac{x}{x - 1} \\
&= \frac{x - 1 - x}{x - 1} \\
&= \frac{-1}{x - 1} \\
f \circ f \circ f(x) &= f[f \circ f(x)] \\
&= f\left(\frac{-1}{x - 1}\right) \\
&= 1 - \frac{1}{\frac{-1}{x-1}} \\
&= 1 + x - 1 \\
&= x \\
\therefore f \circ f \circ f(x) &= x
\end{aligned}$$

Definition 31 (Linear Function) *A linear function is a function of the form $y = f(x) = mx + c$, where m and c are constants, and graph of $y = mx + c$ is always straight line.*

For example, the equation $y = 5x + 1$ is the equation of straight line

Definition 32 (Quadratic Function) *The function $f(x) = ax^2 + bx + c$, $a \neq 0$ is known as quadratic function. The graph of this function is the parabola.*

Definition 33 (Implicit Function) *A two variable function is of the form $f(x, y) = 0$, is called implicit function.*

For example: $x^2 + y^2 = 0$ and $x + 2y = 0$ are implicit functions.

Definition 34 (Explicit Function) *A two variable function is of the form $y = f(x)$ or $x = f(y)$, is called explicit function.*

For example: $x^2 = y^3$ and $y = -3x$ are explicit functions.

Definition 35 (Polynomial Function) *The function $f : R \to R$ defined as $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, is known as Polynomial function of degree n. The different values $3, 4, 5$, etc. of n gives the different functions which are called polynomial functions.*

Definition 36 (Exponential Function) *The function $f : R \to R$ defined as $f(x) = a^x$, $(a > 0)$ and $a \neq 1)$, is known as Exponential function.*

Properties of exponential Function

For any $a, b \in R$ and $x, y \in R$

1. $a^x \cdot a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$
5. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

Definition 37 (Logarithmic Function) *The function $f : R^+ \to R$ defined as $f(x) = log_a x$, is known as logarithmic function.*

1. If $a = e$, $log_e x = lnx$ is known as natural logarithmic.
2. If $a = 10$, $log_1 0x = logx$ is known as common logarithmic.
3. Exponential and logarithmic functions correlated as
   $y = a^x \iff x = log_a y$, $a \in R^{\{}+\} - 1$

Properties of Logarithmic Function

For any $a \in R^+ - 1$ and $x, y \in R^{\{}+\}$ then following axioms are holds,

1. $log_a xy = log_a x + log_a y$

2. $log_a\left(\frac{x}{y}\right) = log_a x - log_a y$

3. $log_a x = \frac{log_b x}{log_b a}$

4. $log_a x^n = n log_a x$

5. $a log_a x = x$, $log_e e^x = 1$

6. $log_a 1 = 0$, $log_a a = 1$

7. $log_e e = 1$

Example 27 *If $f(x) = y = \frac{ax+b}{cx-a}$, prove that $x = f(y)$.*

Solution:

$$
\begin{aligned}
f(x) = y &= \frac{ax+b}{cx-a} \\
\therefore \quad y &= \frac{ax+b}{cx-a} \\
\therefore \quad y(cx-a) &= ax+b \\
\therefore \quad cyx - ay &= ax+b \\
\therefore \quad cyx - ax &= ay+b \\
\therefore \quad x(cy-a) &= ay+b \\
\therefore \quad x &= \frac{ay+b}{cy-a} = f(y)
\end{aligned}
$$

Example 28 *If $g(x) = \frac{x(x-2)}{x-1}$, find $g(0) + g(-1) + g(2) + g(3)$.*

Solution:

$$
\begin{aligned}
g(x) &= \frac{x(x-2)}{x-1} \\
g(-1) &= \frac{(-1)((-1)-2)}{(-1)-1} = \frac{(-1)(-3)}{-2} = \frac{-3}{2} \\
g(0) &= \frac{0(0-2)}{0-1} = 0 \\
g(2) &= \frac{2(2-2)}{2-1} = 0 \\
g(3) &= \frac{3(3-2)}{3-1} = \frac{3}{2} \\
g(0) &+ g(-1) + g(2) + g(3) \\
&= 0 - \frac{3}{2} + \frac{3}{2} + 0 \\
&= 0
\end{aligned}
$$

Example 29 *If* $f(x) = \frac{1}{x+1}$, *prove that* $f(-x) - f(x) = \frac{2x}{1-x^2}$.

Solution:

$$
\begin{aligned}
f(x) &= \frac{1}{x+1} \\
f(-x) &= \frac{1}{-x+1} \\
f(-x) - f(x) &= \frac{1}{1-x} - \frac{1}{x+1} \\
&= \frac{(1+x)-(1-x)}{(1-x)(1+x)} \\
&= \frac{1+x-1+x}{1-x^2} \\
&= \frac{2x}{1-x^2}
\end{aligned}
$$

Hence, proved.

Example 30 *If* $f(x) = log\left(\frac{1+x}{1-x}\right)$, $0 < x < a$; *prove that* $f\left(\frac{2x}{1+x^2}\right) = 2 \cdot f(x)$.

Solution:

$$\begin{aligned}
LHS &= f\left[\frac{2x}{1+x^2}\right] \\
&= log\left[\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right] \\
&= log\left[\frac{\frac{1+x^2+2x}{1+x^2}}{\frac{1+x^2-2x}{1+x^2}}\right] \\
&= log\left[\frac{1+2x+x^2}{1-2x+x^2}\right] \\
&= log\left[\frac{(1+x)^2}{(1-x)^2}\right] \\
&= log\left[\frac{1+x}{1-x}\right]^2 \\
&= 2log\left(\frac{1+x}{1-x}\right) \\
&= 2\cdot f(x) \\
&= RHS
\end{aligned}$$

Example 31 *If* $f = f(p) = 200 - 3p^2$ *is the demand function, then find the demand at* $p = 5$.

Solution:

$$\begin{aligned}
Demand = f(p) &= 200 - 3p^2 \\
f(5) &= 200 - 3(5)^2 \\
&= 200 - 75 \\
&= 125
\end{aligned}$$

2.2 Introduction to Trigonometric Function

Consider a point P on the circle (unit circle). Let θ be the angle between line segment joining to the centre with the positive X-axis. Make a perpendicular line from P on X-axis.

$$sin\theta = \frac{Opposite}{Hypotenus} = \frac{y}{r}$$
$$cos\theta = \frac{Adjecent}{Hypotenus} = \frac{x}{r}$$
$$tan\theta = \frac{sin\theta}{cos\theta} = \frac{y}{x}$$
$$cot\theta = \frac{1}{tan\theta} = \frac{x}{y}$$
$$sec\theta = \frac{1}{cos\theta} = \frac{r}{x}$$
$$cosec\theta = \frac{1}{sin\theta} = \frac{r}{y}$$

These functions $sin\theta$, $cos\theta$, $tan\theta$, $cot\theta$, $sec\theta$ and $cosec\theta$, are trigonometric functions.

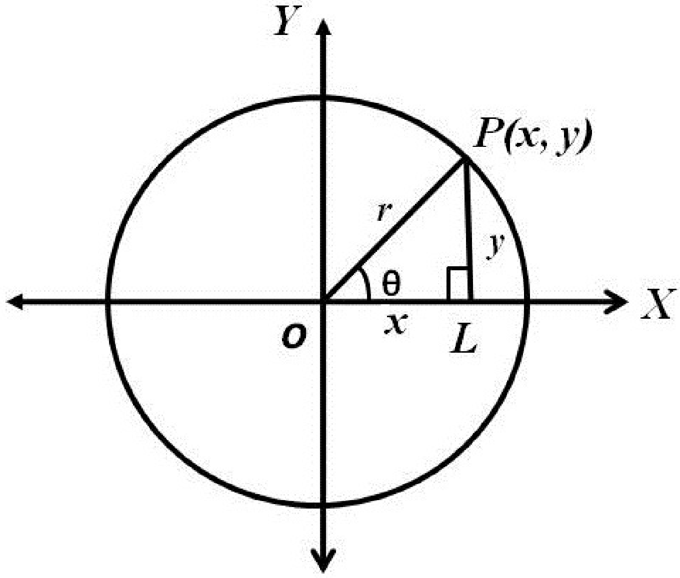


Figure 8: Trigonometric Functions